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SEMILINEAR ELLIPTIC SYSTEMS WITH LACK OF SYMMETRY

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Abstract. By means of a perturbative method introduced by Bolle we give a multiplicity result for a system of semilinear elliptic equations with non–homogeneous boundary conditions in the presence of a generic superquadratic odd nonlinear term.

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1 Introduction

Let $N \geqslant 1$ and $n \geqslant 2$. The main goal of this paper is to prove the existence of multiple solutions $u = (u_1, \ldots, u_N) : \overline{\Omega} \to \mathbb{R}^N$ for the semilinear elliptic system

$$\begin{cases} -\sum\limits_{i,j=1}^{n} \sum\limits_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i u_h) = g_k(x,u) + \varphi_k(x) & \text{in } \Omega \\ u = \chi & \text{on } \partial\Omega \\ k = 1,..,N & \end{cases} \qquad (\mathcal{S}_{\chi,\varphi,N})$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $\varphi = (\varphi_1, \ldots, \varphi_N) \in L^2(\Omega, \mathbb{R}^N)$, $\chi \in H^{1/2}(\partial\Omega, \mathbb{R}^N) \cap C(\partial\Omega, \mathbb{R}^N)$ and the coefficients $a_{ij}^{hk} \in C(\overline{\Omega}, \mathbb{R})$ are such that $a_{ij}^{hk} = a_{ji}^{kh}$. Assume that the Legendre–Hadamard condition holds, i.e., there exists $\nu > 0$ such that

$$\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) \xi_i \xi_j \eta^h \eta^k \;\geqslant\; \nu\, |\xi|^2\, |\eta|^2 \qquad (1.1)$$

for all $x \in \Omega$ and $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^N$. Moreover, suppose that the nonlinear term $g = (g_1, \ldots, g_N) \in C(\overline{\Omega} \times \mathbb{R}^N, \mathbb{R}^N)$ admits a potential G of class C^1 such that

$$\nabla_s G(x,s) = g(x,s)\,, \quad G(x,0) = 0 \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}^N$$

and satisfies the following conditions:

(G_1) there exist $\mu > 2$ and $R > 0$ such that for all $(x,s) \in \Omega \times \mathbb{R}^N$

$$|s| \geqslant R \implies 0 < \mu G(x,s) \leqslant g(x,s) \cdot s \;;$$

(G_2) there exist $\alpha_0 > 0$ and $p > 2$, $p < \frac{2n}{n-2}$ if $n \geqslant 3$, such that

$$|g(x,s)| \leqslant \alpha_0\ (|s|^{p-1} + 1) \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}^N;$$

(G_3) $g(x,-s) = -g(x,s)$ for all $(x,s) \in \Omega \times \mathbb{R}^N$

(here, $\cdot$ denotes the Euclidean scalar product in $\mathbb{R}^N$) .

It is well known that, in the above hypotheses, the problem $(\mathcal{S}_{\chi,\varphi,N})$ has a variational structure and its weak solutions are the critical points of the functional

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \int_\Omega (G(x,u) + \varphi \cdot u)\ dx \tag{1.2}$$

on the manifold $\mathcal{B}_\chi = \{u \in H^1(\Omega, \mathbb{R}^N) :\ u = \chi \text{ a.e. on } \partial\Omega\}$.

Many authors have studied the semilinear elliptic problem

$$\begin{cases} -\Delta u = g(x,u) + \varphi(x) & \text{in } \Omega \\ \quad u = \chi & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

which is a particular case of $(\mathcal{S}_{\chi,\varphi,N})$ with $N = 1$ and $a_{ij}^{hk} = \delta_{ij}^{hk}$.

If $\varphi \equiv \chi \equiv 0$ the problem (1.3) is symmetric, so multiplicity results have been obtained via the equivariant Ljusternik–Schnirelman theory (see, e.g., [12]).

On the contrary, if φ or χ are non–trivial the symmetry is broken and, in general, multiplicity results do not hold .

However, if $\varphi \not\equiv 0$ and $\chi \equiv 0$, in the 80's some perturbative methods have been developed in order to establish the existence of an infinite number of solutions for non–symmetric problems such as $(\mathcal{S}_{0,\varphi,1})$ (see [1, 11, 15, 16]). But these results are partial since an additional assumption needs on the growth of the nonlinearity $g(x,u)$.

In last years the problem (1.3) has been studied also when the boundary condition χ is different from zero. In this case the perturbation term is nonlinear, so the perturbative methods introduced in [1, 11, 15] do not yield a satisfactory result: in particular, in [5] a multiplicity result has been obtained if $g(x,u)$ is homogeneous of type $|u|^{p-2}u$ with $2 < p < 2(1 + \frac{1}{n})$ (see also [6] for another similar result).

More recently, a refined perturbative method introduced by Bolle in [3] and improved in [4] has allowed to better the previous results. In fact, it has been proved the existence of infinitely many solutions of (1.3) when $g(x,u) = |u|^{p-2}u$ for $2 < p < 2(1 + \frac{1}{n-1})$ (cf. [4]).

Both the perturbative approaches used for (1.3) can be extended to the vectorial case $(N > 1)$. In fact, the problem $(\mathcal{S}_{\chi,\varphi,N})$ has been studied in [7] if $g(x,u) = |u|^{p-2}u$ and in [8] only if $a_{ij}^{hk} = \delta_{ij}^{hk}$ and $n \geqslant 3$ but g is not necessarily homogeneous with potential G invariant under the action of a more general group of symmetries.

We point out that such perturbative methods combined with nonsmooth critical point theory allow to study also a class of quasilinear elliptic problems (see [10, 13] and even [14] for a recent result when $n = 2$ and g has an exponential growth).

Here, we consider the system $(\mathcal{S}_{\chi,\varphi,N})$ with a more general function g. Under the previous assumptions, we can state our main results.

Theorem 1.1 *Assume that μ and p satisfy*

$$\frac{\mu}{\mu - p + 1} < \frac{2p}{n(p-2)} . \tag{$\star$}$$

Then, $(\mathcal{S}_{\chi,\varphi,N})$ has a sequence $(u^m)_m \subset \mathcal{B}_\chi$ of solutions with $f(u^m) \to +\infty$.

In particular, if (G_1) holds with $\mu = p$, we obtain the following result.

Corollary 1.2 *Assume that $\mu = p < 2\frac{n+1}{n}$. Then, $(\mathcal{S}_{\chi,\varphi,N})$ admits a sequence $(u^m)_m \subset \mathcal{B}_\chi$ of solutions with $f(u^m) \to +\infty$.*

At last, if $\chi \equiv 0$, weakening the condition $(\star)$ the same result in Theorem 1.1 can be achieved, thus extending the results stated if $N = 1$ in [2, 16] and if $a_{ij}^{hk} = \delta_{ij}^{hk}$ in [8, Theorem 3] to more general elliptic systems.

Theorem 1.3 *Assume that $\chi \equiv 0$ and let μ and p satisfy*

$$\frac{\mu}{\mu - 1} < \frac{2p}{n(p-2)} . \tag{1.4}$$

Then, $(\mathcal{S}_{0,\varphi,N})$ has a sequence $(u^m)_m$ of solutions in $H_0^1(\Omega, \mathbb{R}^N)$ such that $f(u^m) \to +\infty$.

Let us point out that Bolle's perturbative method seems not to allow an improvement of the condition (1.4) if $\chi \equiv 0$. On the contrary, there is a gap between the hypothesis $(\star)$ we need in Theorem 1.1 and the corresponding one in [8, Theorem 2] obtained in the particular case $a_{ij}^{hk} = \delta_{ij}^{hk}$ via [4, Lemma 4.2]. Thus, we think that:

Conjecture 1.4 *Theorem 1.1 (and Corollary 1.2, too) holds true provided that*

$$2 < p < \frac{2n}{n-1}, \quad \partial\Omega \in C^2, \quad \chi \in C^2(\partial\Omega, \mathbb{R}^N), \quad \varphi \in C(\overline{\Omega}, \mathbb{R}^N)$$

and the coefficients a_{ij}^{hk} are sufficiently smooth.

2 Bolle's perturbation arguments

In order to apply the method introduced by Bolle for dealing with problems with broken symmetry, let us recall the main theorem as stated in [4].

The idea is to consider a continuous path of functionals starting from a symmetric functional J_0 and to prove a preservation result for min–max critical levels in order to get critical points also for the end–point functional J_1 (which is the "true" functional of the non–symmetric problem).

Let H be a Hilbert space equipped with the norm $\|\cdot\|$. Assume that $H = H_- \oplus H_+$, where $\dim(H_-) < +\infty$, and let $(e_l)_{l\geqslant 1}$ be an orthonormal base of H_+. Consider

$$H_0 = H_-, \quad H_{l+1} = H_l \oplus \mathbb{R}e_{l+1} \text{ if } l \in \mathbb{N};$$

so $(H_l)_l$ is an increasing sequence of finite dimensional subspaces of H.

Let $J : [0,1] \times H \to \mathbb{R}$ be a C^2–functional and, taken any $\theta \in [0,1]$, set $J_\theta = J(\theta, \cdot) : H \to \mathbb{R}$.

For a given $R > 0$ let us set

$$\Gamma = \{\gamma \in C(H,H) : \ \gamma \text{ is odd and } \gamma(u) = u \text{ if } \|u\| \geqslant R\},$$
$$c_l = \inf_{\gamma \in \Gamma} \sup_{u \in H_l} J_0(\gamma(u)).$$

Assume that

(A_1) J satisfies a weaker form of the classical Palais–Smale condition: any $((\theta^m, u^m))_m \subset [0,1] \times H$ such that

$$(J(\theta^m, u^m))_m \text{ is bounded} \quad \text{and} \quad \lim_{m \to +\infty} J'_{\theta^m}(u^m) = 0 \tag{2.1}$$

converges up to subsequences;

(A_2) for any $b > 0$ there exists $C_b > 0$ such that if $(\theta, u) \in [0,1] \times H$ then

$$|J_\theta(u)| \leqslant b \implies \left| \frac{\partial J}{\partial \theta}(\theta, u) \right| \leqslant C_b \, (\|J'_\theta(u)\| + 1)(\|u\| + 1);$$

(A_3) there exist two continuous maps $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ which are Lipschitz continuous with respect to the second variable and such that $\eta_1(\theta, \cdot) \leqslant \eta_2(\theta, \cdot)$. Suppose that if $(\theta, u) \in [0,1] \times H$ then

$$J'_\theta(u) = 0 \implies \eta_1(\theta, J_\theta(u)) \leqslant \frac{\partial J}{\partial \theta}(\theta, u) \leqslant \eta_2(\theta, J_\theta(u)); \tag{2.2}$$

(A_4) J_0 is even and for each finite dimensional subspace W of H it results

$$\lim_{\substack{u \in W \\ \|u\| \to +\infty}} \sup_{\theta \in [0,1]} J(\theta, u) = -\infty \ .$$

For $i \in \{1,2\}$, let $\psi_i : [0,1] \times \mathbb{R} \to \mathbb{R}$ be the solution of the problem

$$\begin{cases} \frac{\partial \psi_i}{\partial \theta}(\theta, s) = \eta_i(\theta, \psi_i(\theta, s)) \\ \psi_i(0, s) = s \ . \end{cases}$$

Note that $\psi_i(\theta, \cdot)$ is continuous, non–decreasing on $\mathbb{R}$ and $\psi_1(\theta, \cdot) \leqslant \psi_2(\theta, \cdot)$. Set

$$\overline{\eta}_1(s) = \sup_{\theta \in [0,1]} |\eta_1(\theta, s)|, \qquad \overline{\eta}_2(s) = \sup_{\theta \in [0,1]} |\eta_2(\theta, s)| \ .$$

In this framework, the following abstract result can be proved (for more details and the proof, see [3, Theorem 3] and [4, Theorem 2.2]).

Theorem 2.1 *There exists $C \in \mathbb{R}$ such that if $l \in \mathbb{N}$ then*

(a) either J_1 has a critical level $\widetilde{c}_l$ with $\psi_2(1, c_l) < \psi_1(1, c_{l+1}) \leqslant \widetilde{c}_l$,

(b) or $c_{l+1} - c_l \leqslant C \, (\overline{\eta}_1(c_{l+1}) + \overline{\eta}_2(c_l) + 1)$.

Remark 2.2 Let us remark that Theorem 2.1 can be proved also when J_0 is invariant with respect to the action of a more general Lie group of symmetries choosing in a suitable way the sequence of levels $(c_l)_l$ (cf. [8]).

3 Some preliminary lemmas

In order to prove our multiplicity results, first of all we reduce $(\mathcal{S}_{\chi,\varphi,N})$ to an elliptic problem with homogeneous boundary conditions.

Let $\phi \in L^\infty(\Omega, \mathbb{R}^N)$ be the solution of the linear system

$$\begin{cases} \sum\limits_{i,j=1}^{n} \sum\limits_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i \phi_h) = 0 & \text{in } \Omega \\ \phi = \chi & \text{on } \partial\Omega \\ k = 1, .., N. & \end{cases} \tag{3.1}$$

The following result can be readily shown.

Proposition 3.1 *A function $u \in \mathcal{B}_\chi$ is a solution of $(\mathcal{S}_{\chi,\varphi,N})$ if and only if $v \in H_0^1(\Omega, \mathbb{R}^N)$ is a solution of*

$$\begin{cases} -\sum\limits_{i,j=1}^{n} \sum\limits_{h=1}^{N} D_j(a_{ij}^{hk}(x) D_i v_h) = g_k(x, v+\phi) + \varphi_k(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \\ k = 1, .., N & \end{cases}$$

where $u(x) = v(x) + \phi(x)$ for a.e. $x \in \overline{\Omega}$.

Hence, our aim is to state the existence of multiple critical points of the functional

$$J_1(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k \, dx - \int_\Omega (G(x, u+\phi) + \varphi \cdot u) \, dx$$

defined on the Hilbert space $H_0^1(\Omega, \mathbb{R}^N)$ endowed with the scalar product

$$(u, v) = \int_\Omega Du \cdot Dv \, dx = \sum_{k=1}^{N} \int_\Omega \nabla u_k \cdot \nabla v_k \, dx$$

with associated norm $\|\cdot\|$. Moreover, if $1 \leqslant s \leqslant +\infty$, let us denote with $|\cdot|_s$ the usual norm in $L^s(\Omega, \mathbb{R}^N)$.

According to the Bolle's perturbation method, consider the path of functionals $J : [0,1] \times H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ defined as

$$J(\theta, u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k dx - \int_\Omega (G(x, u+\theta\phi) + \theta\varphi \cdot u) dx.$$

Let us remark that it is $J(1, \cdot) = J_1$; so, for simplicity, set $J_\theta = J(\theta, \cdot)$. Clearly, the functional J_0 is even on $H_0^1(\Omega, \mathbb{R}^N)$.

Standard arguments prove that, in our assumptions, J is a C^1–functional and for any $\theta \in [0,1]$ and $u, v \in H_0^1(\Omega, \mathbb{R}^N)$ it is

$$\frac{\partial J}{\partial \theta}(\theta, u) = -\int_\Omega \big(g(x, u+\theta\phi) \cdot \phi \; + \; \varphi \cdot u \big) \, dx$$

and

$$J'_\theta(u)[v] = \frac{\partial J}{\partial u}(\theta, u)[v]$$
$$= \int_\Omega \Big(\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j v_k - g(x, u + \theta\phi) \cdot v - \theta\varphi \cdot v \Big)\ dx.$$

Let us point out that, by integration, the assumption (G_1) implies that there exist $\alpha_1, \alpha_2, \alpha_3 > 0$ such that for all $w \in H_0^1(\Omega, \mathbb{R}^N)$ it is

$$\alpha_1 |w|_\mu^\mu - \alpha_2 \leqslant \int_\Omega G(x, w)\ dx \leqslant \frac{1}{\mu} \int_\Omega g(x, w) \cdot w\ dx + \alpha_3. \tag{3.2}$$

Remark 3.2 The condition $\mu \leqslant p$, which follows by (3.2) and (G_2), can hold together with condition $(\star)$ if $p < 2\left(1 + \frac{1}{n}\right)$. Moreover, $(\star)$ implies

$$p - 1 < \mu. \tag{3.3}$$

The above inequalities allow to state the following results.

Lemma 3.3 *Taken any $\rho \in \,]\frac{1}{\mu}, 1]$ there exist $\beta(\rho), \gamma(\rho) > 0$ such that*

$$J_\theta(u) - \rho J'_\theta(u)[u] \geqslant \left(\frac{1}{2} - \rho\right) \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k\ dx$$
$$+\ \beta(\rho) \int_\Omega |u + \theta\phi|^\mu dx - (1 - \rho)|\varphi|_2 |u|_2 - \gamma(\rho)$$

for any $(\theta, u) \in [0,1] \times H_0^1(\Omega, \mathbb{R}^N)$.

Proof. Let $(\theta, u) \in [0,1] \times H_0^1(\Omega, \mathbb{R}^N)$ and $\rho \in \,]\frac{1}{\mu}, 1]$. By the definition of J_θ it is

$$J_\theta(u) - \rho J'_\theta(u)[u] \ = \ \left(\frac{1}{2} - \rho\right) \int_\Omega \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k\ dx$$
$$- \int_\Omega G(x, u + \theta\phi)\ dx + \rho \int_\Omega g(x, u + \theta\phi) \cdot u\ dx - (1 - \rho)\theta \int_\Omega \varphi \cdot u\ dx.$$

It is quite easy to see that (G_2) and (3.2) imply the existence of some constants $\gamma_1(\rho), \gamma_2(\rho) > 0$ such that

$$- \int_\Omega G(x, u + \theta\phi)\ dx + \rho \int_\Omega g(x, u + \theta\phi) \cdot u\ dx$$
$$\geqslant\ (\rho\mu - 1) \int_\Omega G(x, u + \theta\phi)\ dx - \rho\theta \int_\Omega g(x, u + \theta\phi) \cdot \phi\ dx -\ \gamma_1(\rho)$$
$$\geqslant (\rho\mu - 1)\ \alpha_1 |u + \theta\phi|_\mu^\mu - \rho\alpha_0 \int_\Omega |u + \theta\phi|^{p-1} |\phi|\ dx - \gamma_2(\rho).$$

Let us point out that, taken any $\epsilon > 0$ and a corresponding $\beta(\epsilon) > 0$, (3.3) and the Young's inequality imply

$$|u + \theta\phi|^{p-1}|\phi| \leqslant \epsilon|u + \theta\phi|^{\mu} + \beta(\epsilon)|\phi|^{\frac{\mu}{\mu-p+1}} \qquad \text{for all } x \in \Omega.$$

So, choosing ϵ small enough, by integrating it follows that there exist $\beta(\rho)$ and $\gamma(\rho)$ positive constants such that

$$-\int_{\Omega} G(x, u + \theta\phi)\ dx + \rho \int_{\Omega} g(x, u + \theta\phi) \cdot u\ dx \geqslant \beta(\rho)|u + \theta\phi|_{\mu}^{\mu} - \gamma(\rho);$$

hence, the proof is complete. ■

Remark 3.4 Taken ν as in (1.1), for each $\varepsilon \in\ ]0, \nu[$ there exists $c_{\varepsilon} \geqslant 0$ such that for all $u \in H_0^1(\Omega, \mathbb{R}^N)$ the following Gårding type inequality holds:

$$\int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k\ dx \geqslant (\nu - \varepsilon)\|u\|^2 - c_{\varepsilon}|u|_2^2$$

(see [9, Theorem 6.5.1]). So, for a suitable choice of a positive constant d, for all $u \in H_0^1(\Omega, \mathbb{R}^N)$ we obtain

$$\int_{\Omega} \Big(\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k + d|u|^2 \Big)\ dx \geqslant \frac{\nu}{2}\ \|u\|^2. \tag{3.4}$$

Lemma 3.5 *Let $((\theta^m, u^m))_m \subset [0,1] \times H_0^1(\Omega, \mathbb{R}^N)$ be such that (2.1) holds. Then $((\theta^m, u^m))_m$ converges up to subsequences.*

Proof. If $((\theta^m, u^m))_m$ is such that (2.1) holds, then a constant $K > 0$ exists such that for any $\rho \in\ ]\frac{1}{\mu}, 1]$ Lemma 3.3 implies

$$\begin{aligned} K + \rho\ \|u^m\| \ &\geqslant\ J_{\theta^m}(u^m) - \rho J'_{\theta^m}(u^m)[u^m] \\ &\geqslant\ \left(\frac{1}{2} - \rho\right) \int_{\Omega} \sum_{i,j=1}^{n} \sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h^m D_j u_k^m\ dx \\ &\quad + \beta(\rho) \int_{\Omega} |u^m + \theta\phi|^{\mu} dx - (1-\rho)|\varphi|_2 |u^m|_2 - \gamma(\rho) \end{aligned}$$

for all m large enough. Then, taken d as in (3.4) and $\rho < \frac{1}{2}$, simple calculations imply the existence of $\widetilde{\gamma}(\rho) > 0$ such that

$$\begin{aligned} &K + \rho\ \|u^m\| + \left(\frac{1}{2} - \rho\right) d|u^m|_2^2 + (1-\rho)|\varphi|_2|u^m|_2 \\ &\geqslant\ \left(\frac{1}{2} - \rho\right)\ \frac{\nu}{2}\ \|u^m\|^2 + \frac{\beta(\rho)}{2^{\mu-1}}|u^m|_{\mu}^{\mu} - \widetilde{\gamma}(\rho); \end{aligned}$$

hence, $(u^m)_m$ is bounded in $H_0^1(\Omega, \mathbb{R}^N)$.
Now, taken $\omega^m = J'_{\theta^m}(u^m) + g(x, u^m + \theta^m\phi) + \theta^m\varphi$, it is easy to see that $(\omega^m)_m$ converges strongly in $H^{-1}(\Omega, \mathbb{R}^N)$, up to subsequences. So, arguing as in [7, Lemma 4.1], $(u^m)_m$ has a converging subsequence in $H_0^1(\Omega, \mathbb{R}^N)$. ■

The following results state that the assumptions (A_2) – (A_4) introduced in Section 2 are verified, too.

Lemma 3.6 *For any $b > 0$ there exists $C_b > 0$ such that*

$$|J_\theta(u)| \leqslant b \Longrightarrow \left|\frac{\partial J}{\partial \theta}(\theta, u)\right| \leqslant C_b(\|J'_\theta(u)\| + 1)(\|u\| + 1)$$

for each $(\theta, u) \in [0,1] \times H^1_0(\Omega, \mathbb{R}^N)$.

Proof. Fix $b > 0$ and let $(\theta, u) \in [0,1] \times H^1_0(\Omega, \mathbb{R}^N)$ be such that $|J_\theta(u)| \leqslant b$. Taking $\rho = \frac{1}{2}$ in Lemma 3.3 we have

$$\begin{aligned} b + \frac{1}{2}\|J'_\theta(u)\| \; \|u\| &\geqslant J_\theta(u) - \frac{1}{2} J'_\theta(u)[u] \\ &\geqslant \beta\left(\frac{1}{2}\right)\; |u + \theta\phi|^\mu_\mu - \frac{1}{2}\; |\varphi|_2 |u|_2 - \gamma\left(\frac{1}{2}\right) \end{aligned}$$

and therefore, since $\mu > 2$, straightforward computations and the Young's inequality imply the existence of a constant $\alpha_4 > 0$ such that

$$|u + \theta\phi|^\mu_\mu \leqslant \alpha_4(\|J'_\theta(u)\| \; \|u\| + 1).$$

On the other hand, (G_2) implies

$$\begin{aligned} \left|\frac{\partial J}{\partial \theta}(\theta, u)\right| &\leqslant \left|\int_\Omega g(x, u + \theta\phi) \cdot \phi \; dx\right| + |\varphi|_2 |u|_2 \\ &\leqslant \alpha_0(|u + \theta\phi|^{p-1}_{p-1} + |\Omega|)|\phi|_\infty + |\varphi|_2 |u|_2. \end{aligned} \tag{3.5}$$

Whence, the conclusion follows by the above inequalities and (3.3). ∎

Lemma 3.7 *If $(\theta, u) \in [0,1] \times H^1_0(\Omega, \mathbb{R}^N)$ is such that $J'_\theta(u) = 0$ then the inequality (2.2) holds with $\eta_1, \eta_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = \widetilde{C}\left(s^2 + 1\right)^{\frac{p-1}{2\mu}}$$

for a suitable constant $\widetilde{C} > 0$.

Proof. Let $(\theta, u) \in [0,1] \times H^1_0(\Omega, \mathbb{R}^N)$ be such that $J'_\theta(u) = 0$. By Lemma 3.3 if it is $\rho = \frac{1}{2}$ we obtain

$$J_\theta(u) \geqslant \beta\left(\frac{1}{2}\right)\; |u + \theta\phi|^\mu_\mu - \frac{1}{2}\; |\varphi|_2 |u|_2 - \gamma\left(\frac{1}{2}\right);$$

so, arguing as in the proof of Lemma 3.6, $\sigma > 0$ exists such that

$$|u + \theta\phi|^\mu_\mu \leqslant \sigma(J^2_\theta(u) + 1)^{\frac{1}{2}}.$$

Whence, the conclusion follows by (3.5) and suitable estimates of $|u + \theta\phi|^{p-1}_{p-1}$ and $|u|_2$ with respect to $|u + \theta\phi|^{p-1}_\mu$. ∎

Remark 3.8 If in $(\mathcal{S}_{\chi,\varphi,N})$ the boundary condition χ is identically zero, it is possible to prove the previous lemma with

$$-\eta_1(\theta,s) = \eta_2(\theta,s) = \widetilde{C}\left(s^2+1\right)^{\frac{1}{2\mu}}.$$

Indeed, in this case the path of functionals becomes

$$J(\theta,u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n}\sum_{h,k=1}^{N} a_{ij}^{hk}(x) D_i u_h D_j u_k dx - \int_\Omega (G(x,u)+\theta\varphi\cdot u)dx;$$

hence, $J'_\theta(u) = 0$ implies

$$\left|\frac{\partial J}{\partial\theta}(\theta,u)\right| = \left|\int_\Omega \varphi\cdot u\ dx\right| \leqslant |\varphi|_2|u|_2 \leqslant \widetilde{C}(J_\theta^2(u)+1)^{\frac{1}{2\mu}}.$$

Lemma 3.9 *If W is a finite dimensional subspace of $H_0^1(\Omega,\mathbb{R}^N)$ then*

$$\lim_{\substack{u\in W\\ \|u\|\to+\infty}} \sup_{\theta\in[0,1]} J(\theta,u) = -\infty\ .$$

Proof. It is enough remarking that by (3.2) some positive constants β_1, β_2, β_3 exist such that

$$J_\theta(u) \leqslant \beta_1\|u\|^2 - \beta_2|u|_\mu^\mu - \beta_3 \quad \text{for all } (\theta,u)\in[0,1]\times H_0^1(\Omega,\mathbb{R}^N). \ \blacksquare$$

4 Proof of the main results

Finally, we can apply Theorem 2.1. To this aim let us introduce a suitable class of min-max values for the even functional J_0.

Let $((\lambda^l,u^l))_l$ be a sequence in $\mathbb{R}\times H_0^1(\Omega,\mathbb{R}^N)$ such that

$$\begin{cases} -\Delta u_k^l = \lambda^l u_k^l & \text{in } \Omega \\ u^l = 0 & \text{on } \partial\Omega, \\ k=1,\dots N, \end{cases}$$

with $(u^l)_l$ orthonormalized. Let us consider the finite dimensional subspaces

$$H_0 := \mathbb{R}u^0; \qquad H_{l+1} := H_l \oplus \mathbb{R}u^{l+1} \quad \text{for any } l\in\mathbb{N}.$$

Defined the set of maps Γ as in Section 2 with $H = H_0^1(\Omega,\mathbb{R}^N)$ and a suitable positive constant $R>0$, for all $l\in\mathbb{N}$ let us consider

$$c_l = \inf_{\gamma\in\Gamma}\sup_{u\in H_l} J_0(\gamma(u)).$$

Clearly, it is $c_l \leqslant c_{l+1}$.

Proof of Theorem 1.1. We claim that condition (b) in Theorem 2.1 can not hold for all l large enough. In fact, if we take η_1, η_2 as in Lemma 3.7, condition (b) in Theorem 2.1 becomes

$$c_{l+1} - c_l \ \leqslant\ \widetilde{C}_1\ \left((c_l)^{\frac{p-1}{\mu}} + (c_{l+1})^{\frac{p-1}{\mu}} + 1\right) \tag{4.1}$$

for a suitable $\widetilde{C}_1 > 0$; hence, if (4.1) holds for all l large enough, then by [1, Lemma 5.3] it follows that there exist $\widetilde{\gamma} > 0$ and $l_0 \in \mathbb{N}$ such that

$$c_l \leqslant \widetilde{\gamma}\, l^{\frac{\mu}{\mu-p+1}} \quad \text{for all } l \geqslant l_0. \tag{4.2}$$

On the other hand, by (3.4) and (G_2) it follows that suitable positive constants $\widetilde{\alpha}_1, \widetilde{\alpha}_2$ exist such that

$$J_\theta(u) \geqslant \frac{\nu}{4}\|u\|^2 - \widetilde{\alpha}_1 |u|_p^p - \widetilde{\alpha}_2 \quad \text{for all } (\theta, u) \in [0,1] \times H_0^1(\Omega, \mathbb{R}^N);$$

whence, arguing as in [16] (see also [1]), it is possible to show that if $n \geqslant 3$ there exist $l_1 \in \mathbb{N}$ and $M > 0$ such that

$$c_l \geqslant M\, l^{\frac{2p}{n(p-2)}} \quad \text{for all } l \geqslant l_1,$$

while if $n = 2$ for every $\varepsilon > 0$ there exist $l_\varepsilon \in \mathbb{N}$ and $M_\varepsilon > 0$ such that

$$c_l \geqslant M_\varepsilon\, l^{\frac{p}{p-2}-\varepsilon} \quad \text{for all } l \geqslant l_\varepsilon.$$

But such estimates are in contradiction with (4.2) in the assumption $(\star)$, so condition (a) in Theorem 2.1 holds for infinitely many $l \in \mathbb{N}$. ∎

Proof of Theorem 1.3. By Remark 3.8 it follows that the estimate (4.1) can be replaced by

$$c_{l+1} - c_l \leqslant \widetilde{C}_1 \left((c_l)^{\frac{1}{\mu}} + (c_{l+1})^{\frac{1}{\mu}} + 1 \right).$$

So, arguing as in the proof of Theorem 1.1, it is

$$c_l \leqslant \widetilde{\gamma}\, l^{\frac{\mu}{\mu-1}} \quad \text{for all } l \text{ large enough.}$$

Whence, condition (a) in Theorem 2.1 holds for infinitely many l if μ and p satisfy the hypothesis (1.4). ∎

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References

[1] A. Bahri, H. Berestycki, A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.* **267** (1981) 1–32.

[2] A. Bahri, P.L. Lions, Morse index of some min–max critical points. I. Applications to multiplicity results, *Comm. Pure Appl. Math.* **41** (1988) 1027–1037.

[3] P. Bolle, On the Bolza Problem, *J. Differential Equations* **152** (1999) 274–288.

[4] P. Bolle, N. Ghoussoub, H. Tehrani, The multiplicity of solutions in non–homogeneous boundary value problems, *Manuscripta Math.* **101** (2000) 325–350.

[5] A.M. Candela, A. Salvatore, Multiplicity results of an elliptic equation with non–homogeneous boundary conditions, *Topol. Methods Nonlinear Anal.* **11** (1998) 1–18.

[6] A.M. Candela, A. Salvatore, Some applications of a perturbative method to elliptic equations with non–homogeneous boundary conditions, *Nonlinear Anal.* in press.

[7] A.M. Candela, A. Salvatore, M. Squassina, Multiple solutions for semilinear elliptic systems with non–homogeneous boundary conditions, *Nonlinear Anal.* in press.

[8] M. Clapp, S. Hernandez–Linares, E. Hernandez–Martinez, Linking–preserving perturbations of symmetric functionals, *J. Differential Equations* in press.

[9] C.B. Morrey, Multiple integrals in the calculus of variations, *Springer*, New York (1966).

[10] S. Paleari, M. Squassina, A multiplicity result for perturbed symmetric quasilinear elliptic systems, *Differential Integral Equations* **14** (2001) 785–800.

[11] P.H. Rabinowitz, Multiple critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.* **272** (1982) 753–769.

[12] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conf. Ser. in Math.* **65** Amer. Math. Soc. Providence, R.I. (1986).

[13] M. Squassina, Existence of multiple solutions for quasilinear diagonal elliptic systems, *Electron J. Differential Equations* **14** (1999) 1–12.

[14] M. Squassina, C. Tarsi, Multiple solutions for quasilinear elliptic equations in $\mathbb{R}^2$ with exponential growth, *Manuscripta Math.* **106** (2001) 315–337.

[15] M. Struwe, Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems, *Manuscripta Math.* **32** (1980) 335–364.

[16] K. Tanaka, Morse indices at critical points related to the Symmetric Mountain Pass Theorem and applications, *Comm. Partial Differential Equations* **14** (1989) 99–128.

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