J. Fixed Point Theory Appl. 10 (2011) 191–195
DOI 10.1007/s11784-011-0048-x
Published online March 31, 2011

Journal of Fixed Point Theory and Applications

On Ekeland's variational principle

Marco Squassina

Abstract. For proper lower semicontinuous functionals bounded from below which do not increase upon polarization, an improved version of Ekeland's variational principle can be formulated in Banach spaces, which provides almost symmetric points.

Mathematics Subject Classification (2010). 35A15, 35B06, 58E05.

Keywords. Ekeland's principle, symmetry, weak and strong slope.

1. Introduction

In the study of nonconvex minimization problems, the idea of looking for a special minimizing sequence with good properties in order to guarantee the convergence towards a minimizer goes back to the work of Hilbert and Lebesgue [6, 7]. In this direction one of the main contributions of the last decades in the calculus of variations was surely provided by Ekeland's variational principle for lower semicontinuous functionals on metric spaces, discovered in 1972 [3, 4]. Since then, it has found a multitude of applications in different fields of nonlinear analysis and turned out to be fruitful in simplifying and unifying the proofs of already known results. We refer the reader to the survey [5] and to the monograph [1] for a discussion on a broad range of applications, including optimization, control and geometry of Banach spaces. The aim of the present note is showing that within a suitable abstract symmetrization framework, under the assumption that the functional does not increase by polarization, the conclusion of the principle can be enriched with the useful information that the existing almost critical point is almost symmetric as well. Furthermore, in the context of the critical point theory for nonsmooth functionals originally developed in [2], the result allows to detect a Palais–Smale sequence (u_h) in the sense of weak slope, which becomes more and more symmetric, as $h \to \infty$, yielding a symmetric minimum point, provided that some compactness is available. The additional feature often gives rise to a compactifying effect through suitable compact embeddings of spaces of symmetric functions. For minimax critical values of C^1 functionals, a similar result has been obtained in [8] under the assumption that the functional enjoys a mountain pass geometry. Recently the whole machinery has been extended by the author in [9] to a class of lower semicontinuous functionals in the framework of [2]. Deriving a symmetric version of Ekeland's variational principle in Banach spaces is easier than obtaining the results of [8, 9], since handling a minimization sequence is of course simpler than managing a minimaxing sequence. On the other hand, due to the great impact of Ekeland's principle in the mathematical literature over the last three decades, the author believes that highlighting the precise statements could reveal useful for various applications. Let us now come to the formulation of the results.

Let X and V be two Banach spaces and $S \subseteq X$. We shall consider two maps $* : S \to S$, $u \mapsto u^*$, the symmetrization map, and $h : S \times \mathcal{H}_* \to S$, $(u, H) \mapsto u^H$, the polarization map, $\mathcal{H}_*$ being a path-connected topological space. We assume, according to [8, Section 2.4], that the following hold:

(1) X is continuously embedded in V;
(2) h is a continuous mapping;
(3) for each $u \in S$ and $H \in \mathcal{H}_*$ it holds that $(u^*)^H = (u^H)^* = u^*$ and $u^{HH} = u^H$;
(4) there exists a sequence $(H_m) \subset \mathcal{H}_*$ such that, for every $u \in S$, $u^{H_1 \cdots H_m}$ converges to u^* in V;
(5) for every $u, v \in S$ and $H \in \mathcal{H}_*$ it holds that $\|u^H - v^H\|_V \le \|u - v\|_V$.

Moreover, the mapping $* : S \to V$ can be extended to $* : X \to V$ by setting $u^* := (\Theta(u))^*$ for every $u \in X$, where $\Theta : (X, \|\cdot\|_V) \to (S, \|\cdot\|_V)$ is a Lipschitz function, of Lipschitz constant $C_\Theta > 0$, such that $\Theta|_S = \mathrm{Id}|_S$. We refer the reader to [8, Section 2.4] for some examples of concrete situations suitable in applications to partial differential equations.

We recall [8, Corollary 3.1] a useful approximation result.

Proposition 1.1. *For all $\rho > 0$ there exists a continuous mapping $\mathbb{T}_\rho : S \to S$ such that the image $\mathbb{T}_\rho u$ is built via iterated polarizations and $\|\mathbb{T}_\rho u - u^*\|_V < \rho$ for every $u \in S$.*

In the above framework, here is the main result.

Theorem 1.2. *Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semicontinuous functional bounded from below such that*

$$f(u^H) \le f(u) \quad \text{for all } u \in S \text{ and } H \in \mathcal{H}_*. \tag{1.1}$$

Let $u \in S$, $\rho > 0$ and $\sigma > 0$ with

$$f(u) < \inf_X f + \rho\sigma.$$

Then there exists $v \in X$ such that

(a) $\|v - v^*\|_V < C\rho$;
(b) $\|v - u\| \le \rho + \|\mathbb{T}_\rho u - u\|$;
(c) $f(v) \le f(u)$;
(d) $f(w) \ge f(v) - \sigma\|w - v\|$ *for all* $w \in X$,

for some positive constant C depending only upon V, X and Θ.

Denoting by $|df|(u)$ the weak slope [2] of f at u (it is $|df|(u) = \|df(u)\|_{X'}$ if f is C^1), we say that $(u_j) \subset X$ is a *symmetric Palais–Smale sequence* at level $c \in \mathbb{R}$ ($(SPS)_c$-sequence) if $|df|(u_j) \to 0$, $f(u_j) \to c$ and, in addition, $\|u_j - u_j^*\|_V \to 0$ as $j \to \infty$. We say that f satisfies the *symmetric Palais–Smale condition* at level c ($(SPS)_c$ in short), if any $(SPS)_c$-sequence has a subsequence converging in X. In this context, we have the following result.

Corollary 1.3. *Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semicontinuous functional bounded from below which satisfies* (1.1)*. Moreover, assume that for all $u \in X$ there exists $\xi \in S$ with $f(\xi) \le f(u)$. Then, for any $\varepsilon > 0$, there exists $v \in X$ such that*

(a) $\|v - v^*\|_V \le C\varepsilon$;
(b) $f(v) < \mathcal{M} + \varepsilon^2$, $\mathcal{M} := \inf f$;
(c) $|df|(v) \le \varepsilon$,

for some $C > 0$. In particular, f has a $(SPS)_{\mathcal{M}}$-sequence. If f satisfies $(SPS)_{\mathcal{M}}$, then f admits a critical point (for the weak slope) $z \in X$ with $f(z) = \mathcal{M}$ and $z = z^*$.

As it will be shown in a forthcoming paper, this result is useful for applications to partial differential equations (cf. also [4, Section 4.B, pp. 335–338]) via the additional control (a) that often gives rise to compactifying effects (see, for instance, the arguments of [8, pp. 479–480]) via suitable compact embeddings of spaces of symmetric functions (we refer the reader, for instance, to [10, Section I.1.5]). Note that the assumption that, for all $u \in X$, there exists an element $\xi \in S$ such that $f(\xi) \le f(u)$ is satisfied in many concrete situations, like when X is a Sobolev space, S is the cone of its positive functions and the functional satisfies $f(|u|) \le f(u)$, for all $u \in X$.

Finally, let $\mathcal{B}^*_{\mathcal{H}_*}$ be the set of $\varphi \in X^*$ such that $\|\varphi\| \le 1$, $\langle \varphi, u \rangle \le \langle \varphi, u^H \rangle$ for $u \in S$ and $H \in \mathcal{H}_*$ and for any $u \in X$ there exists $\xi \in S$ with $\langle \varphi, u \rangle \le \langle \varphi, \xi \rangle$, where X^* is the dual of X. In the spirit of [4, Corollary 2.4], Corollary 1.3 yields the following density result.

Corollary 1.4. *Assume that $f : X \to \mathbb{R}$ is a Gâteaux differentiable function satisfying the assumptions of Corollary* 1.3*. Moreover, let $\alpha > 0$ and $\beta \in \mathbb{R}$ be such that*

$$f(v) \ge \alpha\|v\| + \beta \quad \text{for all } v \in X.$$

Then, if $\mathcal{S} := \{v \in X : \|v - v^*\|_V \le 1\}$, the set $df(\mathcal{S}) \subset X^*$ is dense in $\alpha\mathcal{B}^*_{\mathcal{H}_*}$.

2. Proofs

Proof of Theorem 1.2. Let $u \in S$, $\rho > 0$ and $\sigma > 0$ be such that $f(u) < \inf f + \rho\sigma$. If $\mathbb{T}_\rho : S \to S$ is the continuous mapping of Proposition 1.1, we set $\tilde{u} := \mathbb{T}_\rho u \in S$. Then, by construction we have $\|\tilde{u} - u^*\|_V < \rho$ and, in light of (1.1) and the property that $\tilde{u}$ is built from u through iterated polarizations, we obtain

$$f(\tilde{u}) < \inf_X f + \rho\sigma.$$

By Ekeland's variational principle (cf. [3, 4, 5]), there exists an element $v \in X$ such that

$$f(v) \leq f(\tilde{u}), \quad \|v - \tilde{u}\| \leq \rho, \quad f(w) \geq f(v) - \sigma\|w - v\| \quad \text{for all } w \in X.$$

Hence (d) holds and, since $f(v) \leq f(\tilde{u}) \leq f(u)$, conclusion (c) follows as well. In the abstract symmetrization framework, it is readily seen (just argue as in [8, Proposition 2.6] taking into account that Θ is Lipschitz of constant C_Θ) that $\|u^* - v^*\|_V \leq C_\Theta \|u - v\|_V$ for all $u, v \in X$. Then, if $K > 0$ is the continuity constant of the injection $X \hookrightarrow V$, it follows that

$$\begin{aligned}\|v - v^*\|_V &\leq \|v - \tilde{u}\|_V + \|\tilde{u} - u^*\|_V + \|u^* - v^*\|_V \\ &\leq K(C_\Theta + 1)\|v - \tilde{u}\| + \|\tilde{u} - u^*\|_V < (K(C_\Theta + 1) + 1)\rho,\end{aligned}$$

where we used the fact that $u^* = \tilde{u}^*$, in light of (3) of the abstract framework and, again, by the way $\tilde{u}$ is built from u. Then, also conclusion (a) holds true. Finally, we have

$$\|v - u\| \leq \|v - \tilde{u}\| + \|\tilde{u} - u\| \leq \rho + \|\mathbb{T}_\rho u - u\|, \tag{2.1}$$

yielding (b). This concludes the proof of the theorem. □

Proof of Corollary 1.3. Given $\varepsilon > 0$, let $u \in X$ be such that $f(u) < \mathcal{M} + \varepsilon^2$. By assumption, we can find an element $\hat{u} \in S$ such that

$$f(\hat{u}) < \mathcal{M} + \varepsilon^2. \tag{2.2}$$

Then, we are allowed to apply Theorem 1.2 with $\sigma = \rho = \varepsilon$ and get an element $v \in X$ such that $\|v - v^*\|_V \leq C\varepsilon$ for some positive constant C depending only upon V, X and Θ, $f(v) \leq f(\hat{u})$ and $f(w) \geq f(v) - \varepsilon\|w - v\|$, for all $w \in X$. Whence, by taking into account inequality (2.2), conclusions (a) and (b) of the corollary hold true. Moreover, since

$$\limsup_{w \to v} \frac{f(v) - f(w)}{\|v - w\|} \leq \varepsilon,$$

by the definition of strong slope $|\nabla f|(u)$ (see [2]) it follows that $|\nabla f|(u) \leq \varepsilon$. Hence, since the weak slope satisfies $|df|(u) \leq |\nabla f|(u)$ (see [2]), assertion (c) immediately follows. Choosing now a sequence $(\varepsilon_j) \subset \mathbb{R}^+$ with $\varepsilon_j \to 0$ as $j \to \infty$, by definition one finds a $(SPS)_\mathcal{M}$-sequence $(v_j) \subset X$. If f satisfies $(SPS)_\mathcal{M}$, then there exists a subsequence, that we still denote by (v_j), which converges to some z in X. Hence, via lower semicontinuity, we get

$$\mathcal{M} \leq f(z) \leq \liminf_{j \to \infty} f(v_j) = \mathcal{M}.$$

Since $|df|(v_j) \to 0$ and $f(v_j) \to f(z) = \mathcal{M}$ as $j \to \infty$, by means of [2, Proposition 2.6], it follows that $|df|(z) \leq \liminf_j |df|(v_j) = 0$. Notice that, since $\|v_j - v_j^*\|_V \to 0$ as $j \to \infty$, letting $j \to \infty$ into the inequality

$$\|z - z^*\|_V \leq \|z - v_j\|_V + \|v_j - v_j^*\|_V + \|v_j^* - z^*\|_V \leq (C_\Theta + 1)K\|v_j - z\| + \|v_j - v_j^*\|_V$$

yields $z = z^*$, as desired. This concludes the proof of the corollary. □

Acknowledgment

The author research was partially supported by the (PRIN 2007) research project: Metodi Variazionali e Topologici nello Studio di Fenomeni non Lineari.

The author wishes to thank Prof. Ivar Ekeland for a nice feedback on a preliminary version of the manuscript. The author also thanks the referee for his careful reading of the paper and helpful comments.

References

[1] J.-P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*. Pure and Applied Mathematics, Wiley & Sons, New York, 1984.

[2] M. Degiovanni and M. Marzocchi, *A critical point theory for nonsmooth functionals*, Ann. Mat. Pura Appl. (4) **167** (1994), 73–100.

[3] I. Ekeland, *Sur les problémes variationnels*. C. R. Acad. Sci. Paris Sér. A-B **275** (1972), A1057–A1059.

[4] I. Ekeland, *On the variational principle*. J. Math. Anal. Appl. **47** (1974), 324–353.

[5] I. Ekeland, *Nonconvex minimization problems*. Bull. Amer. Math. Soc. (N.S.) **1** (1979), 443–474.

[6] D. Hilbert, *Uber das Dirichletsche Prinzip*. Jber. Deut. Math. Ver. **8** (1900), 184–188.

[7] H. Lebesgue, *Sur le probléḿs de Dirichlet*. Rend. Circ. Mat. Palermo **24** (1907), 371–402.

[8] J. Van Schaftingen, *Symmetrization and minimax principles*. Commun. Contemp. Math. **7** (2005), 463–481.

[9] M. Squassina, *Radial symmetry of minimax critical points for non-smooth functionals*. Commun. Contemp. Math., to appear.

[10] M. Willem, *Minimax Theorems*. Progress in Nonlinear Differential Equations and Their Applications 24, Birkhäuser, Boston, 1996.

Marco Squassina
Dipartimento di Informatica
Università degli Studi di Verona
Cá Vignal 2, Strada Le Grazie 15
I-37134 Verona
Italy
e-mail: marco.squassina@univr.it