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On the Struwe–Jeanjean–Toland monotonicity trick

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The abstract version of Struwe's monotonicity trick developed by Jeanjean and Toland for functionals depending on a real parameter is strengthened in the sense that it provides, for almost every value of the parameter, the existence of a bounded almost symmetric Palais–Smale sequence at the mountain-pass level whenever a mild symmetry assumption is set on the energy functional. In addition, the whole theory is extended to the case of continuous functionals on Banach spaces, in the framework of non-smooth critical point theory.

1. Introduction

It is known that there are situations, often related to physically relevant partial differential equations (PDEs) associated with an energy functional f, where it is particularly difficult to establish the *boundedness* of *Palais–Smale* sequences for f. In order to overcome this difficulty, Struwe [2, 23–27] introduced, around 1988, the so-called *monotonicity trick*. In solving important problems, he showed how the fact that the underlying functional enjoys some monotonicity properties could be used in order to derive a bounded Palais–Smale sequence. About 10 years later, it was shown by Jeanjean [12] that it was possible to formulate a general abstract statement based upon the monotonicity trick. This contribution is of particular relevance since it provides a ready-to-use result in order to tackle variational PDEs for which the Palais–Smale condition is hard to manage. The principle says, essentially, that, given a family of C^1 smooth functionals $f(\lambda;\cdot)$ satisfying a uniform mountain-pass geometry and monotonically depending on the parameter λ, the almost-everywhere differentiability of the mountain-pass value $c(\lambda)$ induces the existence of a bounded Palais–Smale sequence for $f(\lambda;\cdot)$ for almost every λ in the interval Λ where the family is defined. This property cannot be improved in general, in light of a counter-example due to Brézis and Nirenberg [12], which shows that in some cases there may exist values of λ for which *any* Palais–Smale sequence at the level $c(\lambda)$ is *unbounded*. Similar phenomena are known to occur in the study of periodic solutions to Hamiltonian systems [9, 10]. We refer the reader to [12] for applications to a Landesman–Lazer-type problem on $\mathbb{R}^N$, to [11] for a use in bifurcation analysis and, finally, to [13, 33], where the technique was used to investigate some classes of nonlinear Schrödinger equations. An important extension was given in [14], where it became clear that for the monotonicity trick to hold true neither the monotonicity of the family $f(\lambda;\cdot)$ nor the differentiability of its related mountain-pass value $c(\lambda)$ are actually needed. Although for the majority of concrete problems the dependence of the family $f(\lambda;\cdot)$ upon λ is monotone, in [14] some situations were covered in the case where the family $f(\lambda;\cdot)$ has the form $J(\lambda;u) - \lambda I(u)$, where $I, J\colon X \to \mathbb{R}$ are C^1 functionals with suitable structural assumptions. The abstract results of [12] have also been extended, for example, by Szulkin, Zou and Schechter to other minimax structures with a nice impact on PDEs (see [18, 19, 28, 34] and references therein).

The scope of our paper is twofold.

As our primary goal, in theorem 3.1 and corollaries 3.2 and 3.3, we improve the abstract (C^1) version of the work of Jeanjean and Toland [14] in the sense that, up to a set of null measure, for each value of the parameter λ we can find a bounded Palais–Smale sequence $(u_h) \subset X$ for f at the mountain-pass level $c(\lambda)$ which is *almost symmetric*, in the sense that

$$\|u_h - u_h^*\|_V \to 0 \quad \text{as } h \to \infty, \tag{1.1}$$

where V is a Banach space with $X \hookrightarrow V$ continuously, whenever a symmetry assumption, satisfied for a wide range of concrete cases, is assumed on f. Such sequences will be called $(\mathrm{SBPS})_{c(\lambda)}$-sequences (see definition 2.7). Here u^* denotes an abstract symmetrization of u (according to [31]); for instance, it can be the classical Schwarz symmetrization when we take $X = W_0^{1,p}(\Omega)$ for Ω either a ball in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. If, in addition, the functional satisfies the *symmetric bounded Palais–Smale condition* ($(\mathrm{SBPS})_{c(\lambda)}$), then at the limit one finds a symmetric mountain-pass critical point. We stress that, in various situations (like *non-compact problems*) showing that, for some level $c \in \mathbb{R}$, a functional satisfying $(\mathrm{SBPS})_c$ is possible and quite direct (cf. [31, proof of theorem 4.5]), while the Palais–Smale condition, in general, fails [32, theorem 8.4]. In fact, handling an $(\mathrm{SBPS})_c$ sequence allows us to exploit the *compact* embeddings of a spaces of symmetric functions into a suitable Banach space (see, for example, [32, § 1.5]). In some sense, as also pointed out in [31], the additional information about the *almost symmetry* of the Palais–Smale sequence provides an alternative to *concentration-compactness* [15, 16]. Notice that this means that the energy functional is *not a priori* restricted to a space of symmetric functions, as is usually done in applying the well-known Palais *symmetric criticality principle* [17], recently extended by Squassina [22] to a non-smooth framework (see also [21]).

As a second goal, we shall extend the monotonicity trick to the class of *continuous* functionals, in the framework of non-smooth critical point theory. If $\Omega \subset \mathbb{R}^2$ is bounded, applications of the monotonicity trick have been provided [8, 27] for the problem

$$-\Delta u = \lambda \biggl(\int_\Omega \mathrm{e}^u\biggr)^{-1} \mathrm{e}^u$$

with Dirichlet boundary conditions, which is naturally associated with the C^1 functional $f(\lambda;\cdot)\colon H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(\lambda;u) = \frac{1}{2}\int_\Omega |Du|^2 - \lambda \log\biggl(\frac{1}{|\Omega|}\int_\Omega \mathrm{e}^u\biggr).$$

The above equation can be also studied on a Riemannian manifold (M, g), in which case the Laplace operator is replaced by the Laplace–Beltrami operator Δ_g. More generally, following some indications coming from differential geometry [30], one can think of equations on a manifold, associated with functional having a kinetic part of the form

$$\int_M j(x, s, Ds)\, \mathrm{d}\mu_M(x), \quad j(x, s(x), Ds(x)) = G_{ij}(x, s(x)) D_i s(x) D_j s(x),$$

which, due to the explicit dependence upon $s(x)$ in the integrand, are non-smooth (not even locally Lipschitz). In a similar fashion, in the context of diffusion processes such as heat conduction, explicit dependence of the $s(x)$ in the kinetic part of the functional has to be expected in the case of non-homogeneous and non-isotropic materials (cf. [3, 29]). Therefore, it is reasonable to think that some situations can occur in which the functional $f(\lambda; \cdot)$ under study is of the form $J(\lambda; u) - \lambda I(u)$, where $J(\lambda; \cdot)\colon X \to \mathbb{R}$ are merely continuous (or even less regular) functionals, while $I(\lambda; \cdot)\colon X \to \mathbb{R}$ are C^1 functionals. In order to deal with this level of generality, we shall use a suitable non-smooth critical point theory, developed about 20 years ago (see, for example, [5–7]) and now well established.

The plan of the paper is as follows. In § 2, we recall a few notions and results from non-smooth critical point theory and symmetrization theory. In § 3 we state and comment on the main result of the paper (theorem 3.1) as well as two useful consequences (corollaries 3.2 and 3.3). Finally, in § 4, we provide the proofs of the results.

2. Some preliminary facts

In this section we recall abstract notions and results from non-smooth critical point and symmetrization theories that will be used in the proof of the main results.

2.1. Tools from symmetrization theory

We refer the reader to [31] and references therein.

2.1.1. *Abstract symmetrization*

Let X and V be two Banach spaces and $S \subseteq X$. We consider two maps $*\colon S \to S$, $u \mapsto u^*$ (*symmetrization map*) and $h\colon S \times \mathcal{H}_* \to S$, $(u, H) \mapsto u^H$ (*polarization map*), where $\mathcal{H}_*$ is a path-connected topological space. We assume the following conditions:

(i) X is continuously embedded in V;

(ii) h is a continuous mapping;

(iii) for each $u \in S$ and $H \in \mathcal{H}_*$ it holds that $(u^*)^H = (u^H)^* = u^*$ and $u^{HH} = u^H$;

(iv) there exists $(H_m) \subset \mathcal{H}_*$ such that, for $u \in S$, $u^{H_1 \cdots H_m}$ converges to u^* in V;

(v) for every $u, v \in S$ and $H \in \mathcal{H}_*$ it holds that $\|u^H - v^H\|_V \leqslant \|u - v\|_V$.

Furthermore, $*\colon S \to V$ can be extended to the whole space X by setting $u^* := (\Theta(u))^*$ for all $u \in X$, where $\Theta\colon (X, \|\cdot\|_V) \to (S, \|\cdot\|_V)$ is a Lipschitz function such that $\Theta|_S = \mathrm{Id}\,|_S$. It is readily seen that, within this framework, there exists $C_\Theta > 0$ such that

$$\|u^* - v^*\|_V \leqslant C_\Theta \|u - v\|_V \quad \text{for all } u, v \in X. \tag{2.1}$$

2.1.2. *Concrete polarization*

A subset H of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points x which satisfy $\alpha \cdot x \leqslant \beta$ for some $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ with $|\alpha| = 1$. Given x in $\mathbb{R}^N$ and a polarizer H the reflection of x with respect to the boundary of H is denoted by x_H. The polarization of a function $u\colon \mathbb{R}^N \to \mathbb{R}^+$ by a polarizer H is the function $u^H\colon \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\} & \text{if } x \in H, \\ \min\{u(x), u(x_H)\} & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{2.2}$$

The polarization $C^H \subset \mathbb{R}^N$ of a set $C \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H} = (\chi_C)^H$, where χ denotes the characteristic function. The polarization u^H of a positive function u defined on $C \subset \mathbb{R}^N$ is the restriction to C^H of the polarization of the extension $\tilde{u}\colon \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C. The polarization of a function which may change sign is defined by $u^H := |u|^H$, for any given polarizer H.

2.1.3. *Concrete symmetrization*

The Schwarz symmetrization of a set $C \subset \mathbb{R}^N$ is the unique open ball centred at the origin C^* such that $\mathcal{L}^N(C^*) = \mathcal{L}^N(C)$, where $\mathcal{L}^N$ is the N-dimensional outer Lebesgue measure. If the measure of C is zero we set $C^* = \emptyset$, while if the measure of C is not finite we put $C^* = \mathbb{R}^N$. A measurable function u is admissible for the Schwarz symmetrization if it is non-negative and, for every $\varepsilon > 0$, the Lebesgue measure of $\{u > \varepsilon\}$ is finite. The Schwarz symmetrization of an admissible function $u\colon C \to \mathbb{R}^+$ is the unique function $u^*\colon C^* \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$, it holds that $\{u^* > t\} = \{u > t\}^*$. Considering the extension $\tilde{u}\colon \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C, we have $u^* = (\tilde{u})^*|_{C^*}$ and $(\tilde{u})^*|_{\mathbb{R}^N \setminus C^*} = 0$. The symmetrization for possibly changing sign u can be the defined by $u^* := |u|^*$. Let $\mathcal{H}_* = \{H \in \mathcal{H}\colon 0 \in H\}$ and let Ω be a ball in $\mathbb{R}^N$ or the whole space $\mathbb{R}^N$. Then let us set either

$$X = W_0^{1,p}(\Omega), \quad S = W_0^{1,p}(\Omega, \mathbb{R}^+), \quad V = L^p \cap L^{p^*}(\Omega),$$

or

$$X = S = W_0^{1,p}(\Omega), \quad V = L^p \cap L^{p^*}(\Omega), \quad u^H := |u|^H, \quad u^* := |u|^*.$$

Then (i)–(v) in the abstract framework are satisfied (see, for example, [31]).

2.1.4. *Symmetric approximation of curves*

In the proof of the main result, in order to overcome the lack (in general, cf. [1]) of continuity of the symmetrization map $u \mapsto u^*$, we shall need an approximation tool for continuous curves [31, proposition 3.1] that we adapt to a particular framework. In the following, $\mathbb{D}$ and $\mathbb{S}$ will always denote the closed unit ball and sphere, respectively, in $\mathbb{R}^m$ with $m \geqslant 1$.

PROPOSITION 2.1. *Let X and V be two Banach spaces, $S \subseteq X$, $*$ and $\mathcal{H}_*$ which satisfy the requirements of the abstract symmetrization framework (2.1.1). Let M be a closed subset of $\mathbb{D}$, disjoint from $\mathbb{S}$, and let $\gamma \in C(\mathbb{D}, X)$. Let $H_0 \in \mathcal{H}_*$ and $\gamma(\mathbb{D}) \subset S$. Then, for every $\delta > 0$, there exists a curve $\tilde{\gamma} \in C(\mathbb{D}, X)$ such that*

$$\|\tilde{\gamma}(\tau) - \gamma(\tau)^*\|_V \leqslant \delta \quad \text{for all } \tau \in M, \qquad \tilde{\gamma}(\tau) = \gamma(\tau)^{H_0 H_1 \cdots H_{[\vartheta]} H_\vartheta} \quad \text{for all } \tau \in \mathbb{D},$$

with $\vartheta \geqslant 0$, $H_s \in \mathcal{H}_*$ for $s \geqslant 0$, $\tilde{\gamma}(\tau) = \gamma(\tau)^{H_0}$ for all $\tau \in \mathbb{S}$. Here $[\vartheta]$ denotes the largest integer less than or equal to ϑ and the polarizer H_ϑ is introduced in [31, proposition 2.7].

2.2. Tools from non-smooth critical point theory

For definitions and notions in this section, we refer the reader to [6, 7] and the references therein. In the following, (X, d) will denote a metric space and $B(u, \delta)$ will denote the open ball in X of centre u and of radius δ.

DEFINITION 2.2. Let $f\colon X \to \mathbb{R}$ be a continuous function, and $u \in X$. We denote by $|\mathrm{d}f|(u)$ the supremum of the real numbers σ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H}\colon B(u, \delta) \times [0, \delta] \to X$, such that, for every v in $B(u, \delta)$, and for every t in $[0, \delta]$ we obtain

$$d(\mathcal{H}(v, t), v) \leqslant t, \qquad f(\mathcal{H}(v, t)) \leqslant f(v) - \sigma t.$$

The extended real number $|\mathrm{d}f|(u)$ is called the weak slope of f at u.

We recall from [7] a well-known fact.

PROPOSITION 2.3. *Let $f\colon X \to \mathbb{R}$ be a continuous functional. If $(u_h) \subset X$ is a sequence converging to u in X, then*

$$|\mathrm{d}f|(u) \leqslant \liminf_h |\mathrm{d}f|(u_h).$$

The next result establishes the connection between the weak slope of a function f and its differential $\mathrm{d}f(u)$, in the case where f is of class C^1 [7, corollary 2.12].

PROPOSITION 2.4. *If X is an open subset of a normed space E and f is a function of class C^1 on X, then $|\mathrm{d}f|(u) = \|\mathrm{d}f(u)\|$ for every $u \in X$.*

We recall from [5] the following quantitative deformation lemma [5, theorem 2.3].

LEMMA 2.5. *Assume that X is a complete metric space and $f\colon X \to \mathbb{R}$ is a continuous functional, $c \in \mathbb{R}$, A is a closed subset of X and $\delta, \sigma > 0$ are such that*

$$c - 2\delta \leqslant f(u) \leqslant c + 2\delta \quad \text{and} \quad d(u, A) \leqslant \frac{\delta}{\sigma} \quad \Longrightarrow \quad |\mathrm{d}f|(u) > 2\sigma.$$

Then there exists a continuous map $\eta\colon X \times [0, 1] \to X$ such that

$$d(\eta(u, t), u) \leqslant \frac{\delta}{\sigma} t, \qquad \eta(u, t) \neq u \quad \Longrightarrow \quad f(\eta(u, t)) < f(u),$$

and

$$u \in A, \quad c-\delta \leqslant f(u) \leqslant c+\delta \quad \Longrightarrow \quad f(\eta(u,t)) \leqslant f(u)-(f(u)-c+\delta)t,$$

for every $u \in X$ *and* $t \in [0,1]$.

The previous notions allow us to give the next definition.

DEFINITION 2.6. We say that $u \in \mathrm{dom}(f)$ is a critical point of f if $|\mathrm{d}f|(u)=0$. We say that $c \in \mathbb{R}$ is a critical value of f if there is a critical point $u \in \mathrm{dom}(f)$ of f with $f(u)=c$.

Finally, we consider a useful notion of (almost) symmetry for Palais–Smale sequences.

DEFINITION 2.7. Let $(X,\|\cdot\|)$ and $(V,\|\cdot\|_V)$ be Banach spaces which are compatible with the abstract symmetrization framework 2.1.1. We say that $(u_n) \subset X$ is a *symmetric bounded Palais–Smale sequence* at level $c \in \mathbb{R}$ ($(\mathrm{SBPS})_c$-sequence) if (u_n) is bounded in X, $|\mathrm{d}f|(u_n) \to 0$, $f(u_n) \to c$ and, in addition,

$$\lim_{n\to\infty} \|u_n - u_n^*\|_V = 0.$$

We say that f satisfies the *symmetric bounded Palais–Smale condition* at level c ($(\mathrm{SBPS})_c$ in short), if every $(\mathrm{SBPS})_c$ sequence admits a subsequence converging in X.

3. The results

In this section we state and prove the main results of the paper.

3.1. Assumptions

Let $(X,\|\cdot\|)$ and $(V,\|\cdot\|_V)$ be two real Banach spaces, $S \subseteq X$, $*$ and $\mathcal{H}_*$ which satisfy the requirements of the abstract symmetrization framework (2.1.1). We consider the following assumptions.

$(\mathcal{H}_1)$ Let $\Lambda \subset \mathbb{R}$ be a compact interval and

$$f\colon \Lambda \times X \to \mathbb{R}$$

be a family of functionals such that, for all $\lambda \in \Lambda$, $f(\lambda;\cdot)$ is continuous.

$(\mathcal{H}_2)$ If $\Gamma_0 \subset C(\mathbb{S};X)$, then, for all $\lambda \in \Lambda$,

$$c(\lambda) > a(\lambda), \qquad a(\lambda) := \sup_{\gamma_0\in\Gamma_0}\sup_{\tau\in\mathbb{S}} f(\lambda;\gamma_0(\tau)),$$

where $c(\lambda)$ denotes the mountain-pass values defined by

$$c(\lambda) := \inf_{\gamma\in\Gamma}\sup_{t\in\mathbb{D}} f(\lambda;\gamma(t)), \qquad \Gamma := \{\gamma \in C(\mathbb{D},X)\colon \gamma|_{\mathbb{S}} \in \Gamma_0\}, \quad \Gamma \neq \emptyset. \tag{3.1}$$

$(\mathcal{H}_3)$ For every sequence $(\lambda_h,u_h) \subset \Lambda \times X$ with (λ_h) strictly increasing and converging to λ for which there exists $C \in \mathbb{R}$ with

$$f(\lambda_h;u_h) \leqslant C, \quad -f(\lambda;u_h) \leqslant C, \quad \frac{f(\lambda_h;u_h)-f(\lambda;u_h)}{\lambda-\lambda_h} \leqslant C \quad \text{for all } h \geqslant 1,$$

we have $\|u_h\| \leqslant \mathcal{M}$ for some number $\mathcal{M} = \mathcal{M}(C) \geqslant 0$ and all $h \geqslant 1$ and, for every $\varepsilon > 0$,

$$f(\lambda, u_h) \leqslant f(\lambda_h; u_h) + \varepsilon \quad \text{for all } h \geqslant 1 \text{ sufficiently large.}$$

$(\mathcal{H}_4)$ For all $\gamma \in \Gamma$ there are $\hat{\gamma} \in \Gamma$ and $H_0 \in \mathcal{H}_*$ with $\hat{\gamma}(\mathbb{D}) \subset S$ and $\hat{\gamma}|_{\mathbb{S}}^{H_0} \in \Gamma_0$ such that

$$f(\lambda; \hat{\gamma}(t)) \leqslant f(\lambda; \gamma(t)) \quad \text{for all } t \in \mathbb{D} \text{ and } \lambda \in \Lambda.$$

Moreover, for all $\lambda \in \Lambda$,

$$f(\lambda; u^H) \leqslant f(\lambda; u) \qquad \text{for all } H \in \mathcal{H}_* \text{ and } u \in S.$$

3.1.1. *Some remarks on the assumptions*

Concerning $(\mathcal{H}_2)$, it is a uniform mountain-pass geometry for the family of functions $\{f(\lambda; \cdot)\}_{\lambda \in \Lambda}$. In the minimax principle one could also allow a more general situation where $\Gamma = \Gamma(\lambda)$ depends on λ. On the other hand, in this case one needs some monotonicity property on $\Gamma(\lambda)$, for instance $\Gamma(\lambda) \subseteq \Gamma(\mu)$, for every $\lambda < \mu$. One can recall, for instance, the two important (classical) cases:

$$\Gamma = \{\gamma \in C([0,1]; X) \colon \gamma(0) = 0,\ \gamma(1) = v\} \quad \text{with } f(\lambda; v) < 0 \text{ for all } \lambda \in \Lambda, \tag{3.2}$$

$$\Gamma(\lambda) = \{\gamma \in C([0,1]; X) \colon \gamma(0) = 0,\ f(\lambda; \gamma(1)) < 0\}, \tag{3.3}$$

corresponding in $(\mathcal{H}_2)$ to the choice $\mathbb{D} = [0,1]$, $\mathbb{S} = \{0,1\}$ and $\Gamma_0 = \{0, v\}$. Assuming that the map $\lambda \mapsto f(\lambda; \cdot)$ is decreasing, then $\lambda \mapsto \Gamma(\lambda)$ is increasing. The choice of (3.2) for the construction of $c(\lambda)$ is probably the most classical and widely used, and it is precisely the minimaxing family of curves used in [12,14]. Concerning condition $(\mathcal{H}_3)$, it is precisely the one originally formulated by Jeanjean and Toland [14] and it aims to select a particular sequence (γ_n) of curves in Γ which enjoy some good properties. As pointed out in [14, example 2.1], functionals of the form $f(\lambda; u) = A(\lambda; u) - \lambda B(u)$ satisfy $(\mathcal{H}_3)$, under suitable assumptions. If in addition A is independent of λ, the last property in $(\mathcal{H}_3)$ automatically holds and the boundedness of (u_h) follows by the coerciveness of either $A(u)$ or $B(u)$ [12]. Finally, compared with [14], $(\mathcal{H}_4)$ is the new additional assumption and it constitutes the natural link with symmetrization theory. We stress that it is fulfilled in a broad range of meaningful cases [21, 31]. In the Sobolev case $S = W_0^{1,p}(\Omega, \mathbb{R}^+) \subset W_0^{1,p}(\Omega) = X$ (cf. §§ 2.1.1–2.1.3), choosing the family (3.2), one uses a function $v \geqslant 0$ with $v^{H_0} = v$ and $f(\lambda; v) < 0$ for some $H_0 \in \mathcal{H}_*$ and all $\lambda \in \Lambda$. Hence, if $\gamma \in \Gamma$ and $\hat{\gamma}(t) := |\gamma(t)| \in S$, it follows that $f(\lambda; \hat{\gamma}(t)) \leqslant f(\lambda; \gamma(t))$ for all $t \in [0,1]$ and $\lambda \in \Lambda$ if, for instance, $f(\lambda, |\cdot|) \leqslant f(\lambda, \cdot)$. Moreover, $\hat{\gamma}(0)^{H_0} = 0 \in \Gamma_0$ and $\hat{\gamma}(1)^{H_0} = v^{H_0} = v \in \Gamma_0$. Choosing instead the family (3.3), if we fix some $H_0 \in \mathcal{H}_*$, we have $f(\lambda, \hat{\gamma}(1)^{H_0}) = f(\lambda, |\gamma(1)|^{H_0}) \leqslant f(\lambda, |\gamma(1)|) \leqslant f(\lambda, \gamma(1)) < 0$, so that again $\hat{\gamma}(0)^{H_0}, \hat{\gamma}(1)^{H_0} \in \Gamma_0$, as required by the first part of $(\mathcal{H}_4)$. Similar choices are made in the case where one takes $S = X = W_0^{1,p}(\Omega)$ (cf. §§ 2.1.1–2.1.3).

3.2. Statements

Under $(\mathcal{H}_1)$–$(\mathcal{H}_4)$, we now state the main result of the paper.

Theorem 3.1. *For almost every $\lambda \in \Lambda$, $f(\lambda;\cdot)$ possesses an* $(\mathrm{SBPS})_{c(\lambda)}$*-sequence.*

In turn, under the same hypothesis, we also have the following.

Corollary 3.2. *For almost every $\lambda \in \Lambda$, $f(\lambda;\cdot)$ possesses a critical point $u_\lambda \in X$ at level $c(\lambda)$ and with $u_\lambda = u_\lambda^*$, provided it satisfies* $(\mathrm{SBPS})_{c(\lambda)}$.

Finally, inspired by [12, corollary 1.2], we also have the following.

Corollary 3.3. *Let $f(\lambda;\cdot)$ satisfy $(\mathrm{SBPS})_{c(\lambda)}$ for all $\lambda \in [1-\sigma,1]$, where $\sigma > 0$. Then there exists a sequence $(\lambda_j, u_j) \subset [1-\sigma,1] \times X$ such that $\lambda_j \nearrow 1$ and, for all $j \geqslant 1$,*

$$f(\lambda_j;u_j) = c(\lambda_j), \quad |\mathrm{d}f(\lambda_j;\cdot)|(u_j) = 0, \quad u_j = u_j^*. \tag{3.4}$$

The monotonicity trick in the form of [12, 14] is thus improved in light of the *symmetry* conclusions, as noted in § 1, provided that a symmetry assumption on f, that is $(\mathcal{H}_4)$, is assumed. Corollary 3.3 is particularly useful for the study of the functional $f(1;\cdot)$ on the basis of the properties of the *nearby* functionals $f(\lambda_j;\cdot)$, when bounded Palais–Smale sequences of $f(\lambda;\cdot)$ are precompact for any $\lambda \in [1-\sigma,1]$ (in particular, for $\lambda = 1$). In fact, it is expected that, starting from (3.4) (which imply, in a Sobolev functional framework, that u_j is a symmetric weak solution of an elliptic PDE, possibly in a suitable generalized sense, and thus it is very likely to satisfy *extra qualitative properties*), one can deduce

$$\sup_{j\geqslant 1} \|u_j\| < +\infty,$$

and (in turn, by $u_j \rightharpoonup u$ in X as $j \to \infty$, up to a subsequence)

$$f(1;u_j) \to c(1), \quad |\mathrm{d}f(1;\cdot)|(u_j) \to 0, \quad \text{as } j \to \infty,$$

provided that $\lambda \mapsto c(\lambda)$ is left continuous (cf. [12, lemma 2.3] where this proved in the C^1 case); namely, $f(1;\cdot)$ admits a bounded Palais–Smale sequence at the mountain-pass value $c(1)$. Therefore, by the precompactness of the bounded Palais–Smale sequence for $f(1;\cdot)$, one can conclude that $u_j \to u$ in X as $j \to \infty$, so that $f(1;\cdot)$ admits a non-trivial symmetric ($u = u^*$) critical point u at the mountain-pass level $c(1)$. The symmetry, of course, follows by observing that (on account of (3.4) and (2.1))

$$\|u-u^*\|_V \leqslant \|u-u_j\|_V + \|u_j - u_j^*\|_V + \|u_j^* - u^*\|_V \leqslant 2\|u-u_j\|_V \leqslant C\|u-u_j\|,$$

yielding the desired conclusion, since $u_j \to u$ in X as $j \to \infty$. This line of argument has been successfully followed, without the additional symmetry property, in [13], based upon the monotonicity trick of Jeanjean. Let us also mention that, in a more recent work [4], the authors restrict the functional to a (Sobolev) space X_r of *symmetric functions* in order to recover compactness. With the improved version of the principle given by corollary 3.3, the compactness would be recovered, even working in the full space X, by crucially exploiting the fact that $u_j = u_j^*$ (see (3.4)), which comes from the symmetry of the energy functional. Notice that, in [4], the solution energy level is

$$c_r(1) = \inf_{\gamma\in\Gamma_r} \sup_{t\in[0,1]} f(1;\gamma(t)), \quad \Gamma_r = \{\gamma \in C([0,1],X_r)\colon \gamma(0) = 0,\ f(1;\gamma(1)) < 0\},$$

while, using corollary 3.3, we would find the solution at the level

$$c(1) = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} f(1;\gamma(t)), \quad \Gamma = \{\gamma \in C([0,1], X)\colon \gamma(0) = 0,\ f(1;\gamma(1)) < 0\},$$

thus maintaining the *global minimizing property* of the mountain-pass value.

Finally, theorem 3.1 and corollary 3.2 hold for continuous functionals, in the framework of non-smooth critical point theory, allowing applications to quasi-linear PDEs (cf. [20]).

REMARK 3.4. A possible concrete framework where the abstract results can be applied is the following. Let Ω be either the whole $\mathbb{R}^N$ or the unit ball $B \subset \mathbb{R}^N$ centred at the origin, $N > p \geqslant 2$, $b > a > 0$ and let $f\colon [a,b] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional defined by

$$f(\lambda;u) := \int_\Omega j(u, |\nabla u|) + \frac{1}{p}\int_\Omega |u|^p - \lambda \int_\Omega G(|x|, u), \tag{3.5}$$

where $j \in C^1(\mathbb{R}\times\mathbb{R}^+)$, $t \mapsto j(s,t)$ is strictly convex and increasing and there exist constants $\alpha_0, \alpha_1 > 0$ such that $\alpha_0 t^p \leqslant j(s,t) \leqslant \alpha_1 t^p$ for all $s \in \mathbb{R}$ and $t \in \mathbb{R}^+$ (see, for example, [20]). Then, if the functions

$$G(|x|, s) = \int_0^s g(|x|, t)\,\mathrm{d}t,$$

$g(|x|, s)$, $j_s(s,t)$ and $j_t(s,t)$ satisfy suitable assumptions, conditions $(\mathcal{H}_1)$-$(\mathcal{H}_4)$ hold. In particular, it holds that

$$f(\lambda; u^H) \leqslant f(\lambda; u) \quad \text{for all } \lambda \in [a,b],\ \text{any } H \in \mathcal{H}_* \text{ and } u \in W_0^{1,p}(\Omega)$$

whenever $r \mapsto g(r,s)$ is decreasing, $j(|s|,t) \leqslant j(s,t)$ and $G(|x|,s) \leqslant G(|x|,|s|)$. Notice that, if the growth of j is weakened into $\alpha_0|\xi|^p \leqslant j(s,|\xi|) \leqslant \alpha(|s|)|\xi|^p$ for some possibly unbounded function $\alpha \in C(\mathbb{R})$, then (3.5) is merely lower semi-continuous from $W_0^{1,p}(\Omega)$ to $\mathbb{R}\cup\{+\infty\}$ for any $\lambda \in [a,b]$. Statements 3.1–3.3 are expected to hold also for *lower semi-continuous* functionals with suitable assumptions [21]. On the other hand, in order to avoid excessive technicalities, we prefer to confine our analysis to the continuous case.

4. Proof of the results

Let $\lambda_0 \in \Lambda$ be such that there exist $Q(\lambda_0) \in \mathbb{R}$ and a strictly increasing sequence (λ_h) converging to λ_0 as $h \to \infty$ and

$$\frac{c(\lambda_h) - c(\lambda_0)}{\lambda_0 - \lambda_h} \leqslant Q(\lambda_0) \quad \text{for all } h \geqslant 1. \tag{4.1}$$

As pointed out in [14], due to a result of Denjoy, the set $D \subseteq \Lambda$ of such points λ_0 is such that $\mathcal{L}^1(\Lambda \setminus D) = 0$, where $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure (for a $\lambda \in \Lambda \setminus D$ we would have Dini's derivatives equal to $D^- c(\lambda_0) = D_- c(\lambda_0) = -\infty$, which is only possible on a set of zero measure).

First we formulate an improvement of [14, lemma 2.1], where the existence of suitable almost symmetric paths in Γ enjoying special properties is obtained. We shall state the result for lower semi-continuous functionals.

LEMMA 4.1. *Assume that* $f\colon \varLambda \times X \to \mathbb{R} \cup \{+\infty\}$ *is a family of lower semi-continuous functionals and that* $(\mathcal{H}_2)$*–*$(\mathcal{H}_4)$ *hold. Let* $\lambda_0 \in \varLambda$ *be such that (4.1) is satisfied and let* (λ_h) *be a related strictly increasing sequence converging to* λ_0*. Then there exist* $\bar{h} \geqslant 1$*, two sequences of paths* $(\gamma_h)_{h\geqslant\bar{h}}, (\tilde{\gamma}_h)_{h\geqslant\bar{h}} \subset \varGamma$ *with* $\gamma_h(\mathbb{D}), \tilde{\gamma}_h(\mathbb{D}) \subset S$*, a sequence* $(M_h)_{h\geqslant\bar{h}}$ *of non-empty closed subsets of* $\mathbb{D}$*, disjoint from* $\mathbb{S}$*, and a positive constant* $\mathcal{M}(\lambda_0)$ *such that*

$$\|\tilde{\gamma}_h(t) - \gamma_h(t)^*\|_V \leqslant \lambda_0 - \lambda_h \quad \text{for all } t \in M_h, \tag{4.2}$$

$$f(\lambda_0; \tilde{\gamma}_h(t)) \geqslant c(\lambda_0) - \lambda_0 + \lambda_h \quad \Longrightarrow \quad \|\tilde{\gamma}_h(t)\| \leqslant \mathcal{M}(\lambda_0) \tag{4.3}$$

for all $h \geqslant \bar{h}$ *and furthermore, for all* $\varepsilon > 0$*, it holds that*

$$\sup_{t\in\mathbb{D}} f(\lambda_0; \tilde{\gamma}_h(t)) \leqslant \sup_{t\in\mathbb{D}} f(\lambda_0; \gamma_h(t)) \leqslant c(\lambda_0) + \varepsilon \tag{4.4}$$

for all $h \geqslant \bar{h}$ *sufficiently large.*

Proof. By the definition of $c(\lambda_h)$, as in [14, lemma 2.1], we can select a sequence $(\varrho_h) \subset \varGamma$ of curves such that, for all $h \geqslant 1$ large,

$$\sup_{t\in\mathbb{D}} f(\lambda_h; \varrho_h(t)) \leqslant c(\lambda_h) + \lambda_0 - \lambda_h. \tag{4.5}$$

In view of $(\mathcal{H}_4)$, up to substituting ϱ_h with $\hat{\varrho}_h$, for all $h \geqslant 1$ we may assume, without loss of generality, that $\varrho_h(\mathbb{D}) \subset S$ and $\varrho_h|_{\mathbb{S}}^{H_0(h)} \in \varGamma_0$, for some polarizer $H_0(h) \in \mathcal{H}_*$. Let now $\vartheta \in C(\mathbb{D}, \mathbb{D})$ be defined by setting $\vartheta(\tau) = \tau|\tau|^{-1}$ for all $\tau \in \mathbb{D} \setminus \mathbb{D}/2$ and $\vartheta(\tau) = 2\tau$ for all $\tau \in \mathbb{D}/2$. Consider now the curve $\gamma_h\colon \mathbb{D} \to X$, defined as $\gamma_h(\tau) := \varrho_h(\vartheta(\tau))$ for all $\tau \in \mathbb{D}$. Then, $\gamma_h \in \varGamma$, $\gamma_h(\mathbb{D}) = \varrho_h(\vartheta(\mathbb{D})) = \varrho_h(\mathbb{D}) \subset S$ and, of course,

$$\sup_{t\in\mathbb{D}} f(\lambda_h; \gamma_h(t)) \leqslant c(\lambda_h) + \lambda_0 - \lambda_h. \tag{4.6}$$

Then, by arguing exactly as in the proof of [14, lemma 2.1(ii)] by $(\mathcal{H}_3)$, for all $\varepsilon > 0$,

$$\sup_{t\in\mathbb{D}} f(\lambda_0; \gamma_h(t)) \leqslant c(\lambda_0) + \varepsilon \tag{4.7}$$

for every $h \geqslant 1$ large enough. In view of assumption $(\mathcal{H}_2)$, there exists $\omega = \omega(\lambda_0) > 0$ small enough that $c(\lambda_0) - 3\omega > a(\lambda_0)$. Let us set

$$M_h := \overline{(f(\lambda_0; \cdot) \circ \gamma_h)^{-1}([c(\lambda_0) - 3\omega, c(\lambda_0) + \omega])}. \tag{4.8}$$

Therefore, $M_h \subset \mathbb{D}$ is of course closed and non-empty (just take $\varepsilon = \omega$ in (4.7) and use the definition of $c(\lambda_0)$) for $h \geqslant \bar{h}$, for some $\bar{h} = \bar{h}(\omega) \geqslant 1$. Moreover, $M_h \cap \mathbb{S} = \emptyset$ for all $h \geqslant \bar{h}$. In fact, assume by contradiction that, for some $h \geqslant \bar{h}$, there exists $\tau_h \in M_h \cap \mathbb{S}$. In turn, by definition, there exists a sequence $\xi_j^h \subset \mathbb{D}$ with $\xi_j^h \to \tau_h \in \mathbb{S}$ as $j \to \infty$ and

$$c(\lambda_0) - 3\omega \leqslant f(\lambda_0; \varrho_h(\vartheta(\xi_j^h))) \leqslant c(\lambda_0) + \omega$$

for all $j \geqslant 1$. Then, noticing that $\vartheta(\xi_j^h) \in \mathbb{S}$ for $j \geqslant 1$ sufficiently large by the definition of ϑ, we can conclude that

$$c(\lambda_0) - 3\omega \leqslant f(\lambda_0; \varrho_h(\vartheta(\xi_j^h))) \leqslant \sup_{\tau\in\mathbb{S}} f(\lambda_0; \varrho_h(\tau)) \leqslant a(\lambda_0) < c(\lambda_0) - 3\omega,$$

yielding the desired contradiction. Then, on account of proposition 2.1, for every $h \geqslant \bar{h}$, there exists a curve $\tilde{\gamma}_h \in C(\mathbb{D}, X)$ with $\tilde{\gamma}_h(\mathbb{D}) \subset S$ such that $\|\tilde{\gamma}_h(t) - \gamma_h(t)^*\|_V \leqslant \lambda_0 - \lambda_h$ for all $t \in M_h$ and $\tilde{\gamma}_h(\tau) = \gamma_h(\tau)^{H_0(h)}$ for all $\tau \in \mathbb{S}$. In particular, (4.2) holds. Furthermore, we have $\tilde{\gamma}_h \in \varGamma$, since

$$\tilde{\gamma}_h|_{\mathbb{S}} = \gamma_h|_{\mathbb{S}}^{H_0(h)} = \varrho_h|_{\mathbb{S}}^{H_0(h)} \in \varGamma_0.$$

Taking into account how $\tilde{\gamma}_h$ is constructed (by iterated polarizations, according to lemma 2.1), by assumption $(\mathcal{H}_4)$ and inequality (4.6), for all $h \geqslant \bar{h}$ we have

$$\sup_{t \in \mathbb{D}} f(\lambda_h; \tilde{\gamma}_h(t)) \leqslant \sup_{t \in \mathbb{D}} f(\lambda_h; \gamma_h(t)) \leqslant c(\lambda_h) + \lambda_0 - \lambda_h. \tag{4.9}$$

At this point, proceeding exactly as in the proof of [14, lemma 2.1(i)] there exists a positive constant $\mathcal{M} = \mathcal{M}(\lambda_0)$ such that implication (4.3) holds. Finally, by combining (4.7) with $f(\lambda_0; \tilde{\gamma}_h(t)) \leqslant f(\lambda_0; \gamma_h(t))$ (again in light of $(\mathcal{H}_4)$) it also follows that (4.4) holds. $\square$

We can now proceed with the proof of the main result, theorem 3.1.

4.1. Proof of theorem 3.1

Fix an arbitrary $\lambda_0 \in \varLambda$ such that condition (4.1) is satisfied and let (λ_h) be a related strictly increasing sequence converging to λ_0. We know that the set $D \subseteq \varLambda$ of such values has full measure $\mathcal{L}^1(\varLambda)$. According to lemma 4.1, there exist $\bar{h} \geqslant 1$ (depending upon λ_0), two sequences of paths $(\gamma_h)_{h \geqslant \bar{h}}, (\tilde{\gamma}_h)_{h \geqslant \bar{h}} \subset \varGamma$ with $\gamma_h(\mathbb{D}), \tilde{\gamma}_h(\mathbb{D}) \subset S$, a sequence $(M_h)_{h \geqslant \bar{h}}$ of non-empty closed subsets of $\mathbb{D}$, disjoint from $\mathbb{S}$, and a positive constant $\mathcal{M}(\lambda_0)$ such that conditions (4.2)–(4.4) hold. Let $\omega = \omega(\lambda_0)$ be the positive number which appears in the definition (4.8) of M_h. Then, for any fixed $\delta \in (0, \omega]$ small, there exists $h_\delta \geqslant \bar{h}$ such that the following facts hold:

$$\sup_{t \in \mathbb{D}} f(\lambda_0; \tilde{\gamma}_{h_\delta}(t)) \leqslant \sup_{t \in \mathbb{D}} f(\lambda_0; \gamma_{h_\delta}(t)) \leqslant c(\lambda_0) + \delta, \quad 0 < \lambda_0 - \lambda_{h_\delta} \leqslant \delta, \tag{4.10}$$

$$f(\lambda_0; \tilde{\gamma}_{h_\delta}(t)) \geqslant c(\lambda_0) - \lambda_0 + \lambda_{h_\delta} \quad \Longrightarrow \quad \|\tilde{\gamma}_{h_\delta}(t)\| \leqslant \mathcal{M}(\lambda_0), \tag{4.11}$$

$$\|\tilde{\gamma}_{h_\delta}(t) - \gamma_{h_\delta}(t)^*\|_V \leqslant \delta \quad \text{for all } t \in M_{h_\delta}. \tag{4.12}$$

For all $\delta \in (0, \omega]$, we denote by A_δ the closed set defined as follows:

$$A_\delta := \{u \in X \colon \|u\| \leqslant \mathcal{M}(\lambda_0),\ u \in \tilde{\gamma}_{h_\delta}(\mathbb{D}) \cap f(\lambda_0; \cdot)^{-1}([c(\lambda_0) - 2\delta, c(\lambda_0) + 2\delta])\},$$

and we set

$$C_\delta := \{u \in X \colon d(u, A_\delta) \leqslant \sqrt{\delta},\ c(\lambda_0) - 2\delta \leqslant f(\lambda_0, u) \leqslant c(\lambda_0) + 2\delta\}.$$

Since $f(\lambda_0; \cdot)$ is continuous, C_δ is of course closed in X. We claim that $C_\delta \neq \emptyset$ for any $\delta \in (0, \omega]$. In fact, let $w_\delta := \tilde{\gamma}_{h_\delta}(t_\delta) \in S$ with $t_\delta \in \mathbb{D}$, by continuity, such that

$$\max_{t \in \mathbb{D}} f(\lambda_0; \tilde{\gamma}_{h_\delta}(t)) = f(\lambda_0; w_\delta).$$

Then, it follows that

$$c(\lambda_0) - 2\delta \leqslant c(\lambda_0) - \lambda_0 + \lambda_{h_\delta} \leqslant c(\lambda_0) \leqslant f(\lambda_0; w_\delta) \leqslant c(\lambda_0) + 2\delta.$$

This, by virtue of (4.11), also yields $\|w_\delta\| = \|\tilde{\gamma}_{h_\delta}(t_\delta)\| \leqslant \mathcal{M}(\lambda_0)$. Hence, $w_\delta \in A_\delta$ and, in turn, $w_\delta \in C_\delta$, proving the claim. Now, given $\delta \in (0, \omega]$, assume by contradiction that

$$\text{for all } u \in X, \quad u \in C_\delta \quad \Longrightarrow \quad |\mathrm{d}f(\lambda_0; \cdot)|(u) > 2\sqrt{\delta}. \tag{4.13}$$

By the quantitative deformation lemma (lemma 2.5), applied to $f(\lambda_0; \cdot)$ with the choice $\sigma := \sqrt{\delta}$, we can find a continuous map $\eta_\delta \colon X \times [0,1] \to X$ with the following properties:

$$f(\lambda_0; \eta_\delta(u,t)) \leqslant f(\lambda_0; u), \qquad \|\eta_\delta(u,t) - u\| \leqslant \sqrt{\delta} t, \tag{4.14}$$

$$u \in A_\delta, \quad c(\lambda_0) - \delta \leqslant f(\lambda_0, u) \leqslant c(\lambda_0) + \delta \quad \Longrightarrow \quad f(\lambda_0; \eta_\delta(u,1)) \leqslant c(\lambda_0) - \delta, \tag{4.15}$$

for all $u \in X$ and $t \in [0,1]$. Let now $\Theta \colon X \to [0,1]$ be a continuous function such that

$$\begin{aligned} &\Theta(u) = 0 \quad \text{for all } u \in C_1, \qquad C_1 := \{u \in X \colon f(\lambda_0; u) \leqslant a(\lambda_0)\}, \\ &\Theta(u) = 1 \quad \text{for all } u \in C_2, \qquad C_2 := \{u \in X \colon f(\lambda_0; u) \geqslant c(\lambda_0) - \delta\}. \end{aligned}$$

Such a map exists since C_1, C_2 are non-empty closed subsets of X and $C_1 \cap C_2 = \emptyset$. Then, we consider the curve $\hat{\gamma} \colon \mathbb{D} \to X$ defined by setting

$$\hat{\gamma}(t) := \eta_\delta(\tilde{\gamma}_{h_\delta}(t), \Theta(\tilde{\gamma}_{h_\delta}(t))) \quad \text{for all } t \in \mathbb{D}.$$

Of course $\hat{\gamma}$ is continuous. Moreover, $\hat{\gamma}|_{\mathbb{S}}$ belongs to Γ_0. In fact, taking $\tau \in \mathbb{S}$, we have

$$f(\lambda_0; \tilde{\gamma}_{h_\delta}(\tau)) \leqslant \sup_{\gamma_0 \in \Gamma_0} \sup_{\tau \in \mathbb{S}} f(\lambda_0; \gamma_0(\tau)) = a(\lambda_0).$$

Then, by the definition and properties of η_δ and Θ, we have

$$\hat{\gamma}(\tau) = \eta_\delta(\tilde{\gamma}_{h_\delta}(\tau), \Theta(\tilde{\gamma}_{h_\delta}(\tau))) = \eta_\delta(\tilde{\gamma}_{h_\delta}(\tau), 0) = \tilde{\gamma}_{h_\delta}(\tau) \quad \text{for every } \tau \in \mathbb{S}.$$

Thus, $\hat{\gamma}$ belongs to Γ. Consider now an arbitrary point $t \in \mathbb{D}$. If it is the case that

$$f(\lambda_0; \tilde{\gamma}_{h_\delta}(t)) \leqslant c(\lambda_0) - (\lambda_0 - \lambda_{h_\delta}),$$

then by the first inequality in (4.14) we have

$$f(\lambda_0; \hat{\gamma}(t)) \leqslant c(\lambda_0) - (\lambda_0 - \lambda_{h_\delta}). \tag{4.16}$$

On the contrary, in the case

$$f(\lambda_0; \tilde{\gamma}_{h_\delta}(t)) > c(\lambda_0) - (\lambda_0 - \lambda_{h_\delta}) \geqslant c(\lambda_0) - \delta,$$

it then follows by (4.11) that $\|\tilde{\gamma}_{h_\delta}(t)\| \leqslant \mathcal{M}(\lambda_0)$, namely, on account of (4.10),

$$\tilde{\gamma}_{h_\delta}(t) \in A_\delta, \quad c(\lambda_0) - \delta \leqslant f(\lambda_0; \tilde{\gamma}_{h_\delta}(t)) \leqslant c(\lambda_0) + \delta,$$

yielding, by virtue of implication (4.15) and the definition of Θ,

$$f(\lambda_0; \hat{\gamma}(t)) = f(\lambda_0; \eta_\delta(\tilde{\gamma}_{h_\delta}(t), 1)) \leqslant c(\lambda_0) - \delta \leqslant c(\lambda_0) - (\lambda_0 - \lambda_{h_\delta}). \tag{4.17}$$

Hence, by combining inequalities (4.16) and (4.17), we conclude that

$$c(\lambda_0) \leqslant \sup_{t\in[0,1]} f(\lambda_0; \hat{\gamma}(t)) \leqslant c(\lambda_0) - (\lambda_0 - \lambda_{h_\delta}) < c(\lambda_0),$$

namely the desired contradiction. Therefore, by choosing $\delta = 1/j$, there exists a sequence $(u_j) \subset X$ ($u_j \in C_j$), contained in the ball centred at the origin and of radius $\mathcal{M}(\lambda_0)+2$, such that $f(\lambda_0; u_j) \to c(\lambda_0)$ as $j \to \infty$, and $|\mathrm{d}f(\lambda_0; \cdot)|(u_j) \to 0$ as $j \to \infty$. At this stage, we have proved that $f(\lambda_0; \cdot)$ admits a bounded Palais–Smale sequence at the mountain-pass value $c(\lambda_0)$. Let now A_j, M_j, γ_j and $\tilde{\gamma}_j$ denote A_δ, M_{h_δ}, γ_{h_δ} and $\tilde{\gamma}_{h_\delta}$, respectively, with $\delta = 1/j$ for $j \geqslant 1/\omega$. We claim that $A_j \subset \tilde{\gamma}_j(M_j)$. If $y \in A_j$, there exists $\tau \in \mathbb{D}$ with $y = \tilde{\gamma}_j(\tau)$ and $c(\lambda_0) - 2/j \leqslant f(\lambda_0; \tilde{\gamma}_j(\tau)) \leqslant c(\lambda_0) + 2/j$, yielding, by $(\mathcal{H}_4)$ and(4.10),

$$c(\lambda_0) - 3\omega \leqslant c(\lambda_0) - \frac{2}{j} \leqslant f(\lambda_0; \tilde{\gamma}_j(\tau)) \leqslant f(\lambda_0; \gamma_j(\tau)) \leqslant c(\lambda_0) + \frac{1}{j} \leqslant c(\lambda_0) + \omega.$$

Hence, $\tau \in (f(\lambda_0; \cdot) \circ \gamma_j)^{-1}([c(\lambda_0) - 3\omega, c(\lambda_0) + \omega]) \subset M_j$, namely $y \in \tilde{\gamma}_j(M_j)$, proving the claim. Hence, from $d(u_j, A_j) \leqslant 1/\sqrt{j}$ (recall that $u_j \in C_j$), we deduce

$$d(u_j, \tilde{\gamma}_j(M_j)) \leqslant \frac{1}{\sqrt{j}}. \tag{4.18}$$

u_j^* is defined in § 2.1.1. Moreover, for all $\tau \in M_j$, since $\tilde{\gamma}_j(\tau)^* = \gamma_j(\tau)^*$ by construction and (iii) of the framework in § 2.1.1, we have $\|\gamma_j(\tau)^* - u_j^*\|_V \leqslant C_\Theta \|\tilde{\gamma}_j(\tau) - u_j\|_V$, by inequality (2.1). Then, for some constant C, on account of (4.12) and (4.18),

$$\begin{aligned}
\|u_j - u_j^*\|_V &\leqslant \inf_{\tau\in M_j} [\|u_j - \tilde{\gamma}_j(\tau)\|_V + \|\tilde{\gamma}_j(\tau) - \gamma_j(\tau)^*\|_V + \|\gamma_j(\tau)^* - u_j^*\|_V] \\
&\leqslant \inf_{\tau\in M_j} [(1 + C_\Theta)K\|u_j - \tilde{\gamma}_j(\tau)\| + \|\tilde{\gamma}_j(\tau) - \gamma_j(\tau)^*\|_V] \\
&\leqslant \frac{C}{\sqrt{j}},
\end{aligned}$$

where $K > 0$ is the continuity constant of $X \hookrightarrow V$. This concludes the proof.

4.2. Proof of corollary 3.2

Let $\lambda_0 \in \Lambda$ be such that there exists an $(\mathrm{SBPS})_{c(\lambda_0)}$-sequence $(u_j) \subset X$. Since $f(\lambda_0; \cdot)$ satisfies $(\mathrm{SBPS})_{c(\lambda_0)}$, there exists a subsequence (u_{j_m}) of (u_j) which converges to some u in X. By proposition 2.3, we have $|\mathrm{d}f(\lambda_0; \cdot)|(u) = 0$. By continuity, $f(\lambda_0; u) = c(\lambda_0)$. Recalling (2.1),

$$\begin{aligned}
\|u - u^*\|_V &\leqslant \lim_{j\to\infty} (\|u - u_{j_m}\|_V + \|u_{j_m} - u_{j_m}^*\|_V + \|u_{j_m}^* - u^*\|_V) \\
&\leqslant \lim_{j\to\infty} ((1 + C_\Theta)K\|u - u_{j_m}\| + \|u_{j_m} - u_{j_m}^*\|_V) \\
&= 0,
\end{aligned} \tag{4.19}$$

yielding $u = u^*$, as desired.

4.3. Proof of corollary 3.3

There exists a strictly increasing sequence $(\lambda_j) \subset [1-\sigma, 1]$ converging to 1 such that, for each $j \geqslant 1$, the functional $f(\lambda_j; \cdot)$ admits a symmetric bounded Palais–Smale sequence (u_m^j) at the mountain-pass energy level $c(\lambda_j)$, namely

$$\lim_m f(\lambda_j; u_m^j) = c(\lambda_j), \qquad \lim_m |\mathrm{d}f(\lambda_j; \cdot)|(u_m^j) = 0, \qquad \lim_m \|u_m^j - u_m^{j*}\|_V = 0.$$

Since $f(\lambda_j; \cdot)$ satisfies $(\mathrm{SBPS})_{c(\lambda_j)}$ for all $j \geqslant 1$, there exists a subsequence $(u_{m_k}^j)$ of (u_m^j) such that $u_{m_k}^j \to u_j$ in X as $k \to \infty$. Recalling proposition 2.3, we see that properties (3.4) hold. Notice that the symmetry conclusion follows again as in (4.19).

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