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Fractional logarithmic Schrödinger equations

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By means of nonsmooth critical point theory, we obtain existence of infinitely many weak solutions of the fractional Schrödinger equation with logarithmic nonlinearity. We also investigate the Hölder regularity of the weak solutions.
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1. Introduction

Let $s \in (0,1)$ and $n > 2s$. The nonlinear fractional logarithmic Schrödinger equation

$$\mathrm{i}\phi_t - (-\Delta)^s\phi + \phi \log|\phi|^2 = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^n \tag{1.1}$$

is a generalization of the classical Nonlinear Schrödinger Equation (NLS) with logarithmic nonlinearity [1]. For power type nonlinearities, the fractional Schrödinger equation was derived by Laskin [2–4] by replacing the Brownian motion in the path integral approach with the so called Lévy flights. Although the equation

$$\mathrm{i}\phi_t - \Delta\phi + \phi \log|\phi|^2 = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^n \tag{1.2}$$

has been ruled out as a fundamental quantum wave equation by very accurate experiments on neutron diffraction, it is currently under discussion if this equation can be adopted as a simplified model for some physical phenomena [5–8]. Its relativistic version, with D'Alembert operator in place of the Laplacian, was first proposed in [9] by Rosen. We refer the reader to [1, 10, 11] for existence and uniqueness of solutions of the associated Cauchy problem in a suitable functional framework and to a study of orbital stability, with respect to radial perturbations, of the ground state solution. Although the fractional Laplacian operator $(-\Delta)^s$ and more generally pseudodifferential operators have been a classical topic of functional analysis since long ago, the interest for such operator has constantly increased in the last few years. Nonlocal operators such as $(-\Delta)^s$ naturally arise in continuum mechanics, phase transition phenomena, population dynamics, and game theory, as they are the typical outcome of stochastical stabilization of Lévy processes; see, for example, the work of Caffarelli [12] and the references therein.

In this paper, we aim to study the existence of multiple standing waves solutions to Eq. (1.1), namely $\phi(t,x) = e^{i\omega t}u(x)$, with $\omega \in \mathbb{R}$, where $u \in H^s(\mathbb{R}^n)$ solves the semilinear elliptic problem

$$(-\Delta)^s u + \omega u = u\log u^2 \qquad \text{in } \mathbb{R}^n. \tag{1.3}$$

Without loss of generality, we can restrict to $\omega > 0$, because if u is a solution of Eq. (1.3), then λu with $\lambda \neq 0$ is a solution of $(-\Delta)^s v + (\omega + \log\lambda^2)v = v\log v^2$. From a variational point of view, Eq. 1.3 is formally associated with the functional J on $H^s(\mathbb{R}^n)$ defined by

$$J(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{\omega+1}{2}\int u^2 - \frac{1}{2}\int u^2\log u^2.$$

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The fractional Sobolev space $H^s(\mathbb{R}^n)$ (see [13]) is continuously embedded in $L^q(\mathbb{R}^n)$ for all $2 \le q \le 2_s^*$, where $2_s^* := 2n/(n-2s)$ and its closed subspace $H^s_{\text{rad}}(\mathbb{R}^n)$ is compactly injected in $L^q(\mathbb{R}^n)$ for $2 < q < 2_s^*$ (see [14]). Furthermore, by the fractional logarithmic Sobolev inequality (see [15]), we have

$$\int u^2 \log\left(\frac{u^2}{\|u\|_2^2}\right) + \left(n + \frac{n}{s}\log a + \log \frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}\right)\|u\|_2^2 \le \frac{a^2}{\pi^s}\|(-\Delta)^{s/2}u\|_2^2, \quad a > 0, \tag{1.4}$$

for any $u \in H^s(\mathbb{R}^n)$. Whence, it is easy to see that J satisfies this inequality

$$J(u) \ge \frac{1}{2}\left[\left(1 - \frac{a^2}{\pi^s}\right)\|(-\Delta)^{s/2}u\|_2^2 - \|u\|_2^2 \log\|u\|_2^2 + \left(\omega + 1 + n + \frac{n}{s}\log a + \log\frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}\right)\|u\|_2^2\right], \tag{1.5}$$

for all $u \in H^s(\mathbb{R}^n)$ and $a > 0$ small. However, there are elements $u \in H^s(\mathbb{R}^n)$ such that

$$\int u^2 \log u^2 = -\infty.$$

Thus, in general, the functional fails to be finite as well as of class C^1. On the other hand, it is readily seen that $J : H^s(\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous. For this reason, we use the nonsmooth critical point theory developed by Degiovanni and Zani in [16, 17] for suitable classes of lower semicontinuous functionals, which is based on a generalization of the norm of the differential, the weak slope [18]. We say that $u \in H^s(\mathbb{R}^n)$ is a weak solution to Eq. (1.3) if

$$\int (-\Delta)^{s/2}u(-\Delta)^{s/2}v + \omega\int uv = \int uv\log u^2, \qquad \text{for all } v \in H^s(\mathbb{R}^n) \cap L_c^\infty(\mathbb{R}^n). \tag{1.6}$$

The main result of the paper is the following.

Theorem 1.1
Problem (1.3) admits a sequence of weak solutions $(u_k) \subset H^s_{\text{rad}}(\mathbb{R}^n)$ with $J(u_k) \to +\infty$. Furthermore, $u_k \in C^{0,2s+\sigma}(\mathbb{R}^n)$ for $s < 1/2$ and $u_k \in C^{1,2s-1+\sigma}(\mathbb{R}^n)$ for $s \ge 1/2$, for some $\sigma \in (0, 1)$.

The result extends to the nonlocal case the results obtained in [19] for the existence of multiple bound states $(u_k) \subset H^1_{\text{rad}}(\mathbb{R}^n)$ for Eq. (1.2). Furthermore, it provides Hölder regularity of the solutions depending upon the value of s, following the strategy outlined in [20]. We point out that, differently from [20], the nonlinearity $g(t) = t\log t^2$ extended to zero at $t = 0$ has a very different behavior at the origin because $g(t)/t \to -\infty$ in place of $g(t)/t \to 0$ for $t \to 0$, property which also generates, as described earlier, the loss of smoothness of the functional J over $H^s(\mathbb{R}^n)$. We mention that, in [21], a class of nonautonomous logarithmic Schrödinger equations with one-periodic potentials was recently investigated, and the existence of multiple solutions was obtained by splitting the energy functional into the sum of a C^1 and a convex lower semicontinuous functional and using the critical point theory of [22].

The paper is organized as follows. In Section 2, we collect some preliminary notions and results. In Section 3, we prove that the functional satisfies the Palais–Smale condition in the sense specified in [17]. In Section 4, we prove the existence and the Hölder regularity of the radially symmetric weak solutions.

Throughout the proofs, the letter C, unless explicitly stated, will always denote a positive constant whose value may change from line to line. Moreover, the notation $\int$ will always denote $\int_{\mathbb{R}^n}$.

2. Preliminary results

First, for the sake of self-containedness, we recall the definition of fractional Sobolev space and fractional Laplacian. For any $s \in (0, 1)$, the space $H^s(\mathbb{R}^n)$ is defined as

$$H^s(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) \,:\, \frac{|u(x) - u(y)|}{|x-y|^{n/2+s}} \in L^2(\mathbb{R}^{2n})\right\}$$

and it is endowed with the norm

$$\|u\| := \left(\int |u|^2 + \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\right)^{1/2}.$$

Let $\mathcal{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^n$. We have

Definition 2.1
For any $u \in \mathcal{S}$ and $s \in (0, 1)$, the fractional Laplacian operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = -\frac{1}{2}C(n,s)\int \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}},$$

with

$$C(n,s)=\left(\int \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\right)^{-1}.$$

For functions u with local Hölder continuous derivatives of exponent $\gamma > 2s-1$, the integral defining $(-\Delta)^s u$ exists finite. Observe that, using [13, Proposition 3.6], for every $u, v \in H^s(\mathbb{R}^n)$, we have that

$$\int (-\Delta)^{s/2}u(-\Delta)^{s/2}v = \frac{C(n,s)}{2}\int_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}. \tag{2.1}$$

We now recall some definitions and results of nonsmooth critical point theory by Degiovanni and Zani [17] (see also the references therein). Let $(X, \|\cdot\|_X)$ be a Banach space and $f : X \to \bar{\mathbb{R}}$ be a function. The (critical point) theory we follow is based on a generalized notion of the norm of the derivative, the weak slope. First, we defined it for continuous functions, and then, we extended it for all functions.

Definition 2.2
Let $f : X \to \mathbb{R}$ be continuous and $u \in X$. Then, $|df|(u)$ is the supremum of the σ's in $[0, +\infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0,\delta] \to X$, satisfying

$$d(\mathcal{H}(w,t), w) \le t, \quad f(\mathcal{H}(w,t)) \le f(w) - \sigma t,$$

whenever $w \in B_\delta(u)$ and $t \in [0,\delta]$.

Now, we define the function $\mathcal{G}_f : \mathrm{epi}(f) \mapsto \mathbb{R}$, where $\mathrm{epi}(f) := \{(u,\lambda) \in X \times \mathbb{R} \mid f(u) \le \lambda\}$, by $\mathcal{G}_f(u,\lambda) = \lambda$. If on $X \times \mathbb{R}$, we consider the norm $\|\cdot\|_{X\times\mathbb{R}} = \left(\|\cdot\|_X^2 + |\cdot|^2\right)^{1/2}$ and we denote with $B_\delta(u,\lambda)$ the open ball of center (u,λ) and radius $\delta > 0$, we have that the function $\mathcal{G}_f$ is continuous and Lipschitzian of constant 1, and it allows to generalize the notion of weak slope for noncontinuous functions f as follows (see [23, Proposition 2.3]).

Proposition 2.3
For all $u \in X$ with $f(u) \in \mathbb{R}$, we have

$$|df|(u) = \begin{cases} \frac{|d\mathcal{G}_f|(u,f(u))}{\sqrt{1-|d\mathcal{G}_f|(u,f(u))^2}} & \text{if } |d\mathcal{G}_f|(u,f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(u,f(u)) = 1. \end{cases}$$

This equivalent definition allows us to study the continuous function $\mathcal{G}_f$ instead of the function f. In some cases, it is also useful the notion of equivariant weak slope.

Definition 2.4
Let f be even with $f(0) \in \mathbb{R}$. For every $\lambda \ge f(0)$, we denote $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda)$ the supremum of the σ's in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2) : (B_\delta(0,\lambda) \cap \mathrm{epi}(f)) \times [0,\delta] \to \mathrm{epi}(f)$, satisfying

$$\|\mathcal{H}((w,\mu),t) - (w,\mu)\|_{X\times\mathbb{R}} \le t, \quad \mathcal{H}_2((w,\mu),t) \le \mu - \sigma t, \quad \mathcal{H}_1((-w,\mu),t) = -\mathcal{H}_1((w,\mu),t),$$

whenever $(w,\mu) \in B_\delta(0,\lambda) \cap \mathrm{epi}(f)$ and $t \in [0,\delta]$.

Then we can give the following:

Definition 2.5
Let $c \in \mathbb{R}$. The function f satisfies $(\mathrm{epi})_c$ condition if there exists $\varepsilon > 0$ such that

$$\inf\{|d\mathcal{G}_f|(u,\lambda) \mid f(u) < \lambda, |\lambda - c| < \varepsilon\} > 0.$$

In this framework, we have the following definitions.

Definition 2.6
$u \in X$ is a (lower) critical point of f if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$.

Definition 2.7
Let $c \in \mathbb{R}$. A sequence $\{u_k\} \subset X$ is a Palais–Smale sequence for f at level c if $f(u_k) \to c$ and $|df|(u_k) \to 0$. Moreover f satisfies the Palais-Smale condition at level c if every Palais–Smale sequence for f at level c admits a convergent subsequence in X.

We will apply the following abstract result (see [17, Theorem 2.11]) that is an adaptation of the classical theorem of Ambrosetti–Rabinowitz.

Theorem 2.8
Let X be a Banach space and $f : X \to \bar{\mathbb{R}}$ a lower semicontinuous even functional. Assume that $f(0) = 0$ and there exists a strictly increasing sequence $\{V_k\}$ of finite-dimensional subspaces of X with the following properties:

1. There exist a closed subspace Z of X, $\rho > 0$ and $\alpha > 0$ such that $X = V_0 \oplus Z$ and for every $u \in Z$ with $\|u\|_X = \rho$, $f(u) \ge \alpha$;
2. There exists a sequence $\{R_k\} \subset]\rho, +\infty[$ such that for any $u \in V_k$ with $\|u\|_X \ge R_k$, $f(u) \le 0$;

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3. For every $c \geq \alpha$, the function f satisfies the Palais–Smale condition at level c and $(\mathrm{epi})_c$ condition;
4. $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0,\lambda) \neq 0$, whenever $\lambda \geq \alpha$.

Then there exists a sequence $\{u_k\}$ of critical points of f such that $f(u_k) \to +\infty$.

Of course, here, we need to review some theorems in [17] for the space $H^s(\mathbb{R}^n)$. The following result is useful to prove that our functional satisfies the hypothesis of Theorem 2.8. We know that $H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$. Now, we prove that every function in $H^s(\mathbb{R}^n)$ can be seen as the limit of a particular sequence in $H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$.

Lemma 2.9
For every $v \in H^s(\mathbb{R}^n)$ there exists a sequence $\{v_k\}$ in $H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$ strongly convergent to v in $H^s(\mathbb{R}^n)$ with $-v^- \leq v_k \leq v^+$ a.e. in $\mathbb{R}^n$.

Proof
Assume first $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$. Let $\vartheta_k : \mathbb{R} \to [0,1]$ in $C^{0,1}$ with Lipschitz constant $\lambda_k = C/k$, $\mathrm{supt}(\vartheta_k) \subset [-2k, 2k]$, $\vartheta_k(s) = 1$ on $[-k,k]$. Let us set $v_k := \vartheta_k(v)v$. Then, observe that $v_k(x) \to v(x)$ as $k \to \infty$ and $-v^- \leq v_k \leq v^+$ a.e. in $\mathbb{R}^n$. We have $|v_k(x)| \leq |v(x)|$ and

$$\begin{aligned}|v_k(x) - v_k(y)|^2 &= |(\vartheta_k(v(x)) - \vartheta_k(v(y)))v(x) + (v(x) - v(y))\vartheta_k(v(y))|^2 \\ &\leq 2(C|v(x) - v(y)|^2\|v\|^2_\infty + |v(x) - v(y)|^2) \leq C|v(x) - v(y)|^2.\end{aligned}$$

Whence, $v_k \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$ and, by Lebesgue's theorem, $v_k \to v$ in $H^s(\mathbb{R}^n)$. The general case boils down to the previous case by arguing on $\max\left\{\min\left\{\varphi_j, v^+\right\}, -v^-\right\}$ in place of v, where, by density, $\varphi_j \in C^\infty_c(\mathbb{R}^n)$ converges strongly to v in $H^s(\mathbb{R}^n)$. □

Remark 2.10
Arguing as in the proof of Lemma 2.9, we can get that, for every $u \in H^s_{\mathrm{loc}}(\mathbb{R}^n)$,

$$H^s_{\mathrm{loc}}(\mathbb{R}^n) := \left\{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n) : \|(-\Delta)^{s/2}u\|_2 < +\infty\right\},$$

and $v \in H^s(\mathbb{R}^n)$, there exists a sequence $\{v_k\} \subset V_u$,

$$V_u := \left\{w \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n) : u \in L^\infty\left(\{x \in \mathbb{R}^n : w(x) \neq 0\}\right)\right\},$$

strongly convergent to v in $H^s(\mathbb{R}^n)$ with $-v^- \leq v_k \leq v^+$ a.e. (see also [16, Theorem 2.3]).

Usually, it is not easy to compute the weak slope of a function. Thus, it is often useful to work with a subdifferential, for which calculus rules hold.

Definition 2.11
For all $u \in X$ with $f(u) \in \mathbb{R}$, $v \in X$ and $\epsilon > 0$, we denote by f^0_ϵ the infimum of $r \in \mathbb{R}$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{V} : (B_\delta(u, f(u)) \cap \mathrm{epi}(f)) \times]0,\delta] \to B_\epsilon(v),$$

such that

$$f(w + t\mathcal{V}((w,\mu t)) \leq \mu + rt,$$

whenever $(w,\mu) \in B_\delta(u, f(u)) \cap \mathrm{epi}(f)$ and $t \in]0,\delta]$. Then we define

$$f^0(u; v) := \sup_{\epsilon > 0} f^0_\epsilon(u, v).$$

As shown in [23, Corollary 4.6], the function $f^0(u; \cdot)$ is convex, lower semicontinuous, and positively homogeneous of degree 1. We can now state the definition of the aforementioned subdifferential.

Definition 2.12
For all $u \in X$ with $f(u) \in \mathbb{R}$, we define

$$\partial f(u) = \left\{\alpha \in X' : \langle \alpha, v\rangle \leq f^0(u; v),\ \forall v \in X\right\}.$$

Now, let us define the continuous functions

$$g(s) := \begin{cases} s\log s^2 & s \neq 0 \\ 0 & s = 0 \end{cases} \qquad \text{and} \qquad G(s) := \begin{cases} s^2\log s^2 & s \neq 0 \\ 0 & s = 0 \end{cases}$$

and let

$$f(u) := \frac{1}{2}\int G(u)dx. \tag{2.2}$$

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Note that

$$G(s) = 2\int_0^s (g(t) + t)dt.$$

We have the following preliminary result.

Proposition 2.13
If $u \in H^s_{\rm loc}(\mathbb{R}^n)$, we have that

1. For every $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, $g(u)v \in L^1(\mathbb{R}^n)$;
2. Let $v \in H^s(\mathbb{R}^n)$ and assume that $(g(u)v)^+ \in L^1(\mathbb{R}^n)$ or $(g(u)v)^- \in L^1(\mathbb{R}^n)$, then there exists a sequence $\{v_k\}$ in $H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$ strongly convergent to v in $H^s(\mathbb{R}^n)$ with

$$\lim_{k\to\infty} \int g(u)v_k = \int g(u)v.$$

Proof
If $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, for $\delta \in \left(0, \frac{N+2s}{N-2s}\right)$, we have

$$\begin{aligned}\int |g(u)v| &\le \|v\|_\infty \left(\int_{\mathrm{spt}(v)\cap|u|\le 1} |g(u)| + \int_{\mathrm{spt}(v)\cap|u|>1} |g(u)|\right)\\ &\le C\left(1 + \int_{\mathrm{spt}(v)\cap|u|>1} |u|^{1+\delta}\right) < +\infty,\end{aligned}$$

then we have Eq. (1). To prove Eq. (2), we argue as in [16, Theorem 2.7]. Let us assume for instance that $(g(u)v)^+ \in L^1(\mathbb{R}^n)$ (if $(g(u)v)^- \in L^1(\mathbb{R}^n)$ the proof is similar). By Lemma 2.9, there is a sequence $\{v_k\}$ in $H^s(\mathbb{R}^n)\cap L^\infty_c(\mathbb{R}^n)$ such that $v_k \to v$ in $H^s(\mathbb{R}^n)$ and $-v^- \le v_k \le v^+$ a.e. in $\mathbb{R}^n$ and, by Eq. (1), for every k, $g(u)v_k \in L^1(\mathbb{R}^n)$. But

$$g(u)v_k = g(u)^+v_k - g(u)^-v_k \le g(u)^+v^+ + g(u)^-v^- = (g(u)v)^+ \in L^1(\mathbb{R}^n)$$

and by Fatou's lemma, we have

$$\limsup_k \int g(u)v_k \le \int g(u)v.$$

Hence, if $\int g(u)v = -\infty$, we conclude, otherwise we have, that $g(u)v \in L^1(\mathbb{R}^n)$ because

$$\int |g(u)v| = \int (g(u)v)^+ + \int (g(u)v)^- = 2\int (g(u)v)^+ - \int g(u)v,$$

and $|g(u)v_k| \le |g(u)v|$. Thus, by Lebesgue's theorem we conclude. □

Moreover, we have the following theorem, whose proof is the same of [17, Theorem 3.1].

Theorem 2.14
Let $u \in H^s(\mathbb{R}^n)$ with $f(u) \in \mathbb{R}$. If $\partial(-f)(u) \ne \emptyset$, then

$$\sup\left\{\int (-g(u) - u)v : v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n), \|v\| \le 1\right\} < +\infty,$$

and hence $-g(u) - u \in H^{-s}(\mathbb{R}^n)$ upon identification of $-g(u) - u$ with its unique extension. Furthermore, $\partial(-f)(u) = \{-g(u) - u\}$ and for all $v \in H^s(\mathbb{R}^n)$ with $(g(u)v)^+ \in L^1(\mathbb{R}^n)$ or $(g(u)v)^- \in L^1(\mathbb{R}^n)$, it holds

$$\langle -g(u) - u, v\rangle = \int (-g(u) - u)v.$$

In particular, this holds true for every $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$.

Finally, in our case, the $(\mathrm{epi})_c$ condition and Eq. (3) of Theorem 2.8 is easy to prove thanks to the following theorem.

Theorem 2.15
Let $(u, \lambda) \in \mathrm{epi}(f)$ with $\lambda > f(u)$. Then $|d\mathcal{G}_f|(u, \lambda) = 1$ and, furthermore, $|d_{\mathbb{Z}_2}\mathcal{G}_f|(0, \lambda) = 1$ for all $\lambda > f(0)$.

The proof can be obtained arguing as in [17, Theorem 3.4].

3. Palais–Smale condition

In this section, we prove that J satisfies the Palais–Smale condition, thus we can apply Theorem 2.8 to prove the existence of infinitely many weak solutions to Eq. (1.3), namely functions $u \in H^s(\mathbb{R}^n)$ such that Eq. (1.6) holds for any $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$. Notice that, that if $u \in H^s(\mathbb{R}^n)$ and $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, by Proposition 2.13, we can consider

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$$\langle J'(u), v\rangle = \int (-\Delta)^{s/2}u(-\Delta)^{s/2}v + \omega\int uv - \int uv\log u^2. \tag{3.1}$$

We will need the following:

Proposition 3.1
Let $u \in H^s(\mathbb{R}^n)$ with $J(u) \in \mathbb{R}$ and $|dJ|(u) < +\infty$. Then the following facts hold:

1. $g(u) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap H^{-s}(\mathbb{R}^n)$ and $|\langle \alpha_u, v\rangle| \le |dJ|(u)\|v\|$ for all $v \in H^s(\mathbb{R}^n)$, where

$$\langle \alpha_u, v\rangle := \int (-\Delta)^{s/2}u(-\Delta)^{s/2}v + (\omega+1)\int uv + \langle -g(u) - u, v\rangle. \tag{3.2}$$

   In particular, for every $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$, we have

$$|\langle J'(u), v\rangle| \le |dJ|(u)\|v\|.$$

2. if $v \in H^s(\mathbb{R}^n)$ is such that $(g(u)v)^+ \in L^1(\mathbb{R}^n)$ or $(g(u)v)^- \in L^1(\mathbb{R}^n)$, then $g(u)v \in L^1(\mathbb{R}^n)$ and identity (3.1) holds.

Proof
As in the proof of Eq. (1) in Proposition 2.13, we have $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, we can write our functional as $J(u) = S(u) - f(u)$, where f is as in (2.2) and

$$S(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{\omega+1}{2}\int u^2.$$

Using the properties of the weak slope (see, e.g., [23, Theorem 4.13]), we can see that $\partial J(u) \neq \emptyset$ and, by the calculus rule, $\partial J(u) \subseteq \partial S(u) + \partial(-f)(u)$ (see [23, Corollary 5.3]), the $\partial(-f)(u)$ is nonempty too. By Theorem 2.14, we obtain that $\partial(-f)(u) = \{-u - g(u)\}$. Because S is C^1, again by [23, Corollary 5.3], $\partial S(u) = \{S'(u)\}$ and then by [23, Theorem 4.13, (iii)], we have $\partial J(u) = \{\alpha_u\}$ and

$$|dJ|(u)\|v\| \ge \min\{\|\beta\|_{H^{-s}} : \beta \in \partial J(u)\}\,\|v\| = \|\alpha_u\|_{H^{-s}}\|v\| \ge |\langle \alpha_u, v\rangle|.$$

The second part follows by using Eq. (3.2) and assertion (2) of Proposition 2.13. □

Remark 3.2
It is readily seen that J is lower semicontinuous; see, for example, [19, Proposition 2.2] for the details. Alternatively, one can observe that there exist $q > 2$ and $C > 0$ such that $G(s) \le C|s|^q$ for all $s \in \mathbb{R}$. Then the assertion follows by a variant of Fatou's lemma.

Finally, we can prove the following:

Proposition 3.3
$J\big|_{H^s_{\mathrm{rad}}(\mathbb{R}^n)}$ satisfies the Palais–Smale condition at level c for every $c \in \mathbb{R}$.

Proof
Let $\{u_k\} \subset H^s(\mathbb{R}^n)$ be a Palais–Smale sequence of J, that is, $J(u_k) \to c$ and $|dJ|(u_k) \to 0$; thus, by Proposition 3.1, we have that $\langle J'(u_k), v\rangle = o(1)\|v\|$ for any $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$. It is easy to see that if $u \in H^s(\mathbb{R}^n)$, then $(u^2\log u^2)^+ \in L^1(\mathbb{R}^n)$; thus by Proposition 3.1, u_k is an admissible test function in Eq. 3.1 and

$$\|u_k\|_2^2 = 2J(u_k) - \langle J'(u_k), u_k\rangle \le 2c + o(1)\|u_k\|. \tag{3.3}$$

Using Eq. (1.4), we have that

$$\begin{aligned}\|u_k\|^2 &= 2J(u_k) - \omega\|u_k\|_2^2 + \int u_k^2\log u_k^2\\ &\le 2c + \frac{a^2}{\pi^s}\left\|(-\Delta)^{s/2}u_k\right\|_2^2 + \|u_k\|_2^2\log\|u_k\|_2^2 - \left(\omega + n + \frac{n}{s}\log a + \log\frac{s\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2s}\right)}\right)\|u_k\|_2^2.\end{aligned}$$

Thus, for $a > 0$ and $\delta > 0$ small and by Eq. (3.3), we have

$$\|u_k\|^2 \le C + o(1)\|u_k\|^{1+\delta} + o(1)\|u_k\|$$

and so $\{u_k\}$ is bounded in $H^s(\mathbb{R}^n)$. Let $\{u_k\}$ now be a Palais–Smale sequence for J in $H^s_{\mathrm{rad}}(\mathbb{R}^n)$. By the boundedness of $\{u_k\}$ and thanks to the compact embedding $H^s_{\mathrm{rad}}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ for $2 < p < 2^*_s$, we have that up to a subsequence, there exists $u \in H^s_{\mathrm{rad}}(\mathbb{R}^n)$ such that

$$u_k \rightharpoonup u \text{ in } H^s_{\mathrm{rad}}(\mathbb{R}^n), \qquad u_k \to u \text{ in } L^p(\mathbb{R}^n),\quad 2 < p < 2^*_s, \qquad u_k \to u \text{ a.e. in } \mathbb{R}^n.$$

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We want to prove that for all $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$

$$\int (-\Delta)^{s/2} u (-\Delta)^{s/2} v + \omega \int uv = \int uv \log u^2. \tag{3.4}$$

So fixed $v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$; let us consider $\vartheta_R(u_k)v$, where, given $R > 0$, $\vartheta_R : \mathbb{R} \to [0,1]$ is a $C^{0,1}$ function such that $\vartheta_R(s) = 1$ for $|s| \le R$, $\vartheta_R(s) = 0$ for $|s| \ge 2R$ and $|\vartheta_R'(s)| \le C/R$ in $\mathbb{R}$. Obviously, as in Lemma 2.9, we have that $\vartheta_R(u_k)v \in H^s(\mathbb{R}^n) \cap L^\infty_c(\mathbb{R}^n)$. Thus, by Eqs. (3.1) and (2.1), we have

$$\begin{aligned}
\langle J'(u_k), \vartheta_R(u_k)v\rangle &= \int (-\Delta)^{s/2} u_k (-\Delta)^{s/2}(\vartheta_R(u_k)v) + \omega \int \vartheta_R(u_k) u_k v - \int \vartheta_R(u_k) u_k v \log u_k^2 \\
&= \frac{C(n,s)}{2} \int_{\mathbb{R}^{2n}} \frac{(u_k(x) - u_k(y))(\vartheta_R(u_k(x))v(x) - \vartheta_R(u_k(y))v(y))}{|x-y|^{n+2s}} \\
&\quad + \omega \int \vartheta_R(u_k) u_k v - \int \vartheta_R(u_k) u_k v \log u_k^2 \\
&= \frac{C(n,s)}{2} \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u_k(x))(u_k(x) - u_k(y))}{|x-y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x-y|^{\frac{n+2s}{2}}} \\
&\quad + \frac{C(n,s)}{2} \int_{\mathbb{R}^{2n}} \frac{v(y)(\vartheta_R(u_k(x)) - \vartheta_R(u_k(y)))(u_k(x) - u_k(y))}{|x-y|^{n+2s}} \\
&\quad + \omega \int \vartheta_R(u_k) u_k v - \int \vartheta_R(u_k) u_k v \log u_k^2.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\Bigg| \frac{C(n,s)}{2} \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u_k(x))(u_k(x) - u_k(y))}{|x-y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x-y|^{\frac{n+2s}{2}}} + \omega \int_{\mathbb{R}^n} \vartheta_R(u_k) u_k v - \int_{\mathbb{R}^n} \vartheta_R(u_k) u_k v \log u_k^2 \\
&\quad - \langle J'(u_k), \vartheta_R(u_k)v\rangle \Bigg| \le \|v\|_\infty \frac{C}{R} \int_{\mathbb{R}^{2n}} \frac{|u_k(x) - u_k(y)|^2}{|x-y|^{n+2s}} \le \frac{C}{R}.
\end{aligned}$$

Because

$$\vartheta_R(u_k(x)) \frac{(u_k(x) - u_k(y))}{|x-y|^{\frac{n+2s}{2}}} \text{ is bounded in } L^2(\mathbb{R}^{2n}), \qquad \frac{(v(x) - v(y))}{|x-y|^{\frac{n+2s}{2}}} \in L^2(\mathbb{R}^{2n}),$$

and

$$\vartheta_R(u_k(x)) \frac{(u_k(x) - u_k(y))}{|x-y|^{\frac{n+2s}{2}}} \to \vartheta_R(u(x)) \frac{(u(x) - u(y))}{|x-y|^{\frac{n+2s}{2}}} \quad \text{a.e. } (x,y) \in \mathbb{R}^{2n} \text{ as } k \to +\infty$$

then

$$\int_{\mathbb{R}^{2n}} \frac{(v(x) - v(y))}{|x-y|^{\frac{n+2s}{2}}} \vartheta_R(u_k(x)) \frac{(u_k(x) - u_k(y))}{|x-y|^{\frac{n+2s}{2}}} \to \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u(x))(u(x) - u(y))}{|x-y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x-y|^{\frac{n+2s}{2}}}$$

as $k \to +\infty$. In the same way, taking into account that $\{\vartheta_R(u_k) u_k \log u_k^2\}$ is bounded in $L^2_{\rm loc}(\mathbb{R}^n)$ and because $\vartheta_R(u_k) u_k \log u_k^2 \to \vartheta_R(u) u \log u^2$ a.e. in $\mathbb{R}^n$, we have

$$\left| \frac{C(n,s)}{2} \int_{\mathbb{R}^{2n}} \frac{\vartheta_R(u(x))(u(x) - u(y))}{|x-y|^{\frac{n+2s}{2}}} \frac{(v(x) - v(y))}{|x-y|^{\frac{n+2s}{2}}} + \omega \int \vartheta_R(u) uv - \int \vartheta_R(u) uv \log u^2 \right| \le \frac{C}{R}.$$

Thus, letting $R \to \infty$, (2.1) yields Equation (3.4). Moreover, see again Remark 3.2, we have that

$$\limsup_k \int u_k^2 \log u_k^2 \le \int u^2 \log u^2.$$

Hence, because $\langle J'(u_k), u_k\rangle \to 0$ and choosing $v = u$ in (3.4), we obtain

$$\limsup_k \left(\left\| (-\Delta)^{\frac{s}{2}} u_k \right\|_2^2 + \omega \|u_k\|_2^2 \right) = \limsup_k \int u_k^2 \log u_k^2 \le \int u^2 \log u^2 = \left\| (-\Delta)^{\frac{s}{2}} u \right\|_2^2 + \omega \|u\|_2^2,$$

which implies the convergence of $u_k \to u$ in $H^s_{\rm rad}(\mathbb{R}^n)$. □

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4. Proof of Theorem 1.1

4.1. Proof for existence

To prove the existence of sequence $\{u_k\} \subset H^s(\mathbb{R}^n)$ of weak solutions to Eq. (1.3) with $J(u_k) \to +\infty$, we will apply Theorem 2.8 with $X = H^s_{\rm rad}(\mathbb{R}^n)$. By Proposition 3.3, we know that J satisfies the Palais–Smale condition. Furthermore, by Theorem 2.15, we have that J satisfies $(\text{epi})_c$ and (4) of Theorem 2.8. Hence, we only have to prove that J satisfies also the geometrical assumptions. Obviously, $J(0) = 0$, and by Eq. (1.5), $J(u) \geq c\|u\|^2$, for a suitable a and if $\|u\|_2$ are sufficiently small. Then, if we take $Z = H^s_{\rm rad}(\mathbb{R}^n)$ and $V_0 = \{0\}$, we have (1). Finally, let $\{V_k\}$ be a strictly increasing sequence of finite-dimensional subspaces of $H^s_{\rm rad}(\mathbb{R}^n)$. Because any norm is equivalent on any V_k, if $\{u_m\} \subset V_k$ is such that $\|u_m\| \to +\infty$, then also $\mu_m := \|u_m\|_2 \to +\infty$. Set now $u_m = \mu_m w_m$, where $w_m = \|u_m\|_2^{-1} u_m$. Thus, $\|w_m\|_2 = 1$, $\left\|(-\Delta)^{s/2} w_m\right\|_2 \leq C$ and $\|w_m\|_\infty \leq C$, and so

$$J(u_m) = \frac{\mu_m^2}{2}\left(\left\|(-\Delta)^{s/2} w_m\right\|_2^2 + \omega + 1 - \log \mu_m^2 - \int w_m^2 \log w_m^2\right) \leq \frac{\mu_m^2}{2}\left(C - \log \mu_m^2\right) \to -\infty.$$

Thus, there exist $\{R_k\} \subset]\rho, +\infty[$ such that for $u \in V_k$ with $\|u\| \geq R_k$, $J(u) \leq 0$ and the condition (2) is satisfied.

4.2. Proof for regularity

To prove the regularity, we follow [20]. First of all, we define

$$\mathcal{W}^{\beta,p}(\mathbb{R}^n) = \left\{u \in L^p(\mathbb{R}^n) : \mathcal{F}^{-1}\left[\left(1 + |\xi|^\beta\right)\hat{u}\right] \in L^p(\mathbb{R}^n)\right\}.$$

For the properties of this space, we refer to [20]. Now, let $u \in H^s(\mathbb{R}^n)$ be a solution of Eq. (1.3) and $\{r_i\}$ be a strictly decreasing sequence of positive constants with $r_0 = 1$. Let $B_i = B(0, r_i)$, and define

$$h(x) = u(x) \log u^2(x).$$

We have that

$$|h| \leq C_\delta\left(|u|^{1-\delta} + |u|^{1+\delta}\right) \tag{4.1}$$

for all $\delta \in (0, 1)$. Now, let $\eta_1 \in C^\infty(\mathbb{R}^n)$, $0 \leq \eta_1 \leq 1$, $\eta_1 = 0$ in B_0^c, $\eta_1 = 1$ in $B_{1/2} := B(0, r_{1/2})$ with $r_1 < r_{1/2} < r_0$, and u_1 be the solution of

$$(-\Delta)^s u_1 + \omega u_1 = \eta_1 h \quad \text{in } \mathbb{R}^n$$

namely, $u_1 = \mathcal{K} * (\eta_1 h)$, where $\mathcal{K}(x) = \mathcal{F}^{-1}(1/(\omega + |\xi|^{2s}))$ is the Bessel kernel. Then

$$(-\Delta)^s(u - u_1) + \omega(u - u_1) = (1 - \eta_1)h \quad \text{in } \mathbb{R}^n$$

and so

$$u - u_1 = \mathcal{K} * [(1 - \eta_1)h].$$

By Sobolev embedding theorem, $u \in L^{q_0}(\mathbb{R}^n)$ with $q_0 = 2n/(n - 2s)$. Moreover, by Eq. (4.1), [20, Theorem 3.3] and Hölder inequality, we have that for a.e. $x \in B_1$

$$|u(x) - u_1(x)| \leq C\left(\|\mathcal{K}\|_{L^{s_0}\left(B^c_{r_{1/2}-r_1}\right)}\left\|(1 - \eta_1)^{1/(1-\delta)} u\right\|_{q_0}^{1-\delta} + \|\mathcal{K}\|_{L^{s_1}\left(B^c_{r_{1/2}-r_1}\right)}\left\|(1 - \eta_1)^{1/(1+\delta)} u\right\|_{q_0}^{1+\delta}\right) \tag{4.2}$$

where $s_0 = q_0/(q_0 - 1 + \delta)$, $s_1 = q_0/(q_0 - 1 - \delta)$ and $\delta < \min\{1, (n + 2s)/(n - 2s)\}$. In fact,

$$\begin{aligned}
|u(x) - u_1(x)| &\leq \int_{B^c_{1/2}} |\mathcal{K}(x - y)||(1 - \eta_1(y))h(y)|dy \\
&\leq C\left(\int_{B^c_{1/2}(x)} |\mathcal{K}|^{s_0}\right)^{1/s_0}\left\|(1 - \eta_1)^{1/(1-\delta)} u\right\|_{q_0}^{1-\delta} \\
&\quad + C\left(\int_{B^c_{1/2}(x)} |\mathcal{K}|^{s_1}\right)^{1/s_1}\left\|(1 - \eta_1)^{1/(1+\delta)} u\right\|_{q_0}^{1+\delta}
\end{aligned}$$

and $B^c_{1/2}(x) \subset B^c_{r_{1/2}-r_1}$. Notice that, the same argument shows that for all $z \in \mathbb{R}^n$ and for a.e. $x \in B_1(z)$

$$|u(x) - u_1(x)| \leq C\left(\|\mathcal{K}\|_{L^{s_0}\left(B^c_{r_{1/2}-r_1}\right)}\|u\|_{q_0}^{1-\delta} + \|\mathcal{K}\|_{L^{s_1}\left(B^c_{r_{1/2}-r_1}\right)}\|u\|_{q_0}^{1+\delta}\right).$$

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and thus, in turn, because the right-hand side is independent of the point z, it follows that $u - u_1 \in L^\infty(\mathbb{R}^n)$. Because $u \in L^{q_0}(\mathbb{R}^n)$ and B_0 is bounded, we have that $\eta_1 h \in L^{p_1}(\mathbb{R}^n)$ with $p_1 = q_0/(1+\delta)$. Then $u_1 \in \mathcal{W}^{2s,p_1}(\mathbb{R}^n)$.

If $n < 6s$, then, in Eq. (4.1), we take

$$\delta < \min\left\{1, \frac{6s-n}{n-2s}\right\}$$

and so $p_1 > n/(2s)$.

If $n \geq 6s$, we have that $p_1 < n/(2s)$, and we proceed as follows. By Sobolev embedding and Eq. (4.2), we have that $u \in L^{q_1}(B_1)$ with $q_1 = p_1 n/(n - 2sp_1)$. Then we repeat the procedure, namely, we consider $\eta_2 \in C^\infty(\mathbb{R}^n)$, $0 \leq \eta_2 \leq 1$, $\eta_2 = 0$ in B_1^c, $\eta_1 = 1$ in $B_{3/2} := B(0, r_{3/2})$ with $r_2 < r_{3/2} < r_1$, getting that $u_2 = \mathcal{K} * (\eta_2 h) \in \mathcal{W}^{2s,p_2}(\mathbb{R}^n)$ with $p_2 = q_1/(1+\delta)$.

If $n < 10s$, then in Eq. (4.1), we take

$$\delta < \frac{-(n-4s) + \sqrt{4s(n-s)}}{n-2s}$$

and we have that $p_2 > n/(2s)$.

If $n \geq 10s$, then $p_2 < n/(2s)$, and we iterate this procedure. Straightforward calculations show that

$$\frac{1}{q_{j+1}} = \frac{1}{q_1} + \left(\frac{1}{q_1} - \frac{1}{q_0}\right)\sum_{i=1}^{j}(1+\delta)^i = \frac{(1+\delta)^{j+1}}{q_0} - \frac{2s}{n}\sum_{i=0}^{j}(1+\delta)^i \tag{4.3}$$

and, using Eq. (4.3), that $p_j > n/(2s)$ is equivalent to

$$(n-2s)(1+\delta)^j - 4s\sum_{i=1}^{j-1}(1+\delta)^i - 4s < 0. \tag{4.4}$$

From Eq. (4.4), we get that, if

$$2(2j-1)s \leq n < 2(2j+1)s, \tag{4.5}$$

then we can take δ small enough such that $p_j > n/(2s)$. Of course, this procedure stops in j steps with j that satisfies Eq. (4.5).

Thus, if ℓ is such that $p_\ell > n/(2s)$, because $u_\ell \in \mathcal{W}^{2s,p_\ell}(\mathbb{R}^n)$, by Sobolev imbeddings (see [20, Theorem 3.2]), we have that $u_\ell \in C^{0,\mu}(\mathbb{R}^n)$ for $\mu > 0$ small enough. Moreover, we can estimate $|u - u_\ell|$ in B_ℓ as in Eq. (4.2), and using the smoothness of $\mathcal{K}$ away from the origin (see [20, Theorem 3.3]) and because $|x-y| \geq C > 0$ for $x \in B_\ell$ and $y \in B_{\ell-1/2}^c$, we have that for $x \in B_\ell$

$$|\nabla(u - u_\ell)(x)| \leq \int_{B_{\ell-1/2}^c} |\nabla\mathcal{K}(x-y)|(|u(y)|^{1-\delta} + |u(y)|^{1+\delta}) \leq C(n, s, \|u\|).$$

Then $u - u_\ell \in W^{1,\infty}(B_\ell)$, and so $u - u_\ell \in C^{0,\mu}(B_\ell)$. Then $u \in C^{0,\mu}(B_\ell)$ and the $C^{0,\mu}$-norm depends on $n, s, \|u\|_{H^s}$ and on the finite sequence $r_0, \ldots, r_\ell$. Moving B_ℓ around $\mathbb{R}^n$, we can recover it, obtaining that $u \in C^{0,\mu}(\mathbb{R}^n)$, and because in addition $u \in L^{q_0}(\mathbb{R}^n)$, we get that $u(x) \to 0$ as $|x| \to +\infty$.

We now claim that for all $a, b \in \mathbb{R}$ with $a < b$ and any $\delta \in (0,1)$, we have $g \in C^{0,1-\delta}([a,b])$. Indeed, if $s, t \in [a,b]$ with, for example, $t > s > 0$, then we have

$$|g(t) - g(s)| \leq \int_s^t |g'(\xi)|d\xi \leq 2\int_s^t (|\log\xi| + 1)d\xi \leq C\int_s^t \xi^{-\delta} = C(t^{1-\delta} - s^{1-\delta}) \leq C(t-s)^{1-\delta}.$$

By symmetry, the same inequality holds for negative $s, t \in [a,b]$. If $s, t \in [a,b]$ with, for example, $s \leq 0 \leq t$, we obtain

$$|g(t) - g(s)| \leq |g(t)| + |g(s)| \leq Ct^{1-\delta} + C(-s)^{1-\delta} \leq C(t-s)^{1-\delta},$$

proving the claim. Then the regularity assertions of Theorem 1.1 follow by arguing as in [20, p. 1251].

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References

1. Cazenave T. *An Introduction to Nonlinear Schrödinger Equations*, Textos de Métodos Matemáticos, vol. 26. Universidade Federal do Rio de Janeiro: Rio de Janeiro, 1996.
2. Laskin N. Fractional quantum mechanics. *Physical Review E* 2000; **62**:3135.
3. Laskin N. Fractional quantum mechanics and Levy path integrals. *Physics Letters A* 2000; **268**:298–305.

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4. Laskin N. Fractional Schrödinger equation. *Physical Review E* 2002; **66**:056108.
5. Białynicki-Birula I, Mycielski J. Nonlinear wave mechanics. *Annals of Physics* 1976; **100**:62–93.
6. Białynicki-Birula I, Mycielski J. Gaussons: solitons of the logarithmic Schrödinger equation, special issue on solitons in physics. *Physica Scripta* 1979; **20**:539–544.
7. Białynicki-Birula I, Mycielski J. Wave equations with logarithmic nonlinearities. *Bulletin Academy of Polish Sciences Cl. III* 1975; **23**(461–466).
8. Zloshchastiev KG. Logarithmic nonlinearity in theories of quantum gravity: origin of time and observational consequences. *Gravitation and Cosmology* 2010; **16**:288–297.
9. Rosen G. Dilatation covariance and exact solutions in local relativistic field theories. *Physical Review* 1969; **183**:1186.
10. Cazenave T. Stable solutions of the logarithmic Schrödinger equation. *Nonlinear Analysis* 1983; **7**:1127–1140.
11. Cazenave T, Haraux A. Équations d'évolution avec non linéarité logarithmique. *Annales de la Faculté des Sciences de Toulouse Mathematiques* 1980; **2**:21–51.
12. Caffarelli L. Nonlocal equations, drifts and games. *Nonlinear Partial Differential Equations, Abel Symposia* 2012; **7**:37–52.
13. Di Nezza E, Palatucci G, Valdinoci E. Hitchhiker's guide to the fractional Sobolev spaces. *Bulletin des Sciences Mathematiques* 2012; **136**:521–573.
14. Lions PL. Symétrie et compacité dans les espaces de Sobolev. *Journal of Functional Analysis* 1982; **49**:315–334.
15. Cotsiolis A, Tavoularis N. On logarithmic Sobolev inequalities for higher order fractional derivatives. *Comptes Rendus de l' Academie des Sciences Paris Serie I* 2005; **340**:205–208.
16. Degiovanni M, Zani S. Euler equations involving nonlinearities without growth conditions. *Potential Analysis* 1996; **5**:505–512.
17. Degiovanni M, Zani S. Multiple solutions of semilinear elliptic equations with one-sided growth conditions, nonlinear operator theory. *Mathematical and Computer Modelling* 2000; **32**:1377–1393.
18. Degiovanni M, Marzocchi M. A critical point theory for nonsmooth functionals. *Annali Di Matematica Pura Ed Applicata* 1994; **167**:73–100.
19. d'Avenia P, Montefusco E, Squassina M. On the logarithmic Schrödinger equation. *Communications in Contemporary Mathematics* 2014; **16**:1350032.
20. Felmer P, Quaas A, Tan J. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proceedings of the Royal Society of Edinburg Section A* 2012; **142**:1237–1262.
21. Szulkin A, Squassina M. Multiple solutions to logarithmic Schrödinger equations with periodic potential. *Calculus of Variations and Partial Differential Equations*. to appear.
22. Szulkin A. Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems. *Annales De l Institut Henri Poincaré, Analyse Non Linéaire* 1986; **3**:77–109.
23. Campa I, Degiovanni M. Subdifferential calculus and non-smooth critical point theory. *SIAM Journal of Optimization* 2000; **10**:1020–1048.

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