Zeitschrift für Analysis und ihre Anwendungen
Journal for Analysis and its Applications
Volume 21 (2002), No. 1, 185–201

Asymptotics of Solutions for fully Nonlinear Elliptic Problems at Nearly Critical Growth

A. Musesti and M. Squassina

Abstract. In this paper we deal with the study of limits of solutions of a class of fully nonlinear elliptic problems at nearly critical growth. By means of P.L. Lions' concentration-compactness principle, we prove an alternative result for the existence of non-trivial solutions of the limit problem.

Keywords: *Concentration-compactness principle, critical exponent, best Sobolev constant, fully nonlinear elliptic problems*

AMS subject classification: 35J65, 35B40

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, $1 < p < n$ and $p^* = \frac{np}{n-p}$. In 1989 Guedda and Veron [10] proved that the $p-$Laplacian problem at critical growth

$$\left.\begin{array}{ll} -\Delta_p u = u^{p^*-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array}\right\} \tag{$*$}$$

has no non-trivial solution $u \in W_0^{1,p}(\Omega)$ if the domain Ω is star-shaped. As known, this non-existence result is due to the failure of compactness for the critical Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, which causes a loss of global Palais-Smale condition for the functional associated with problem $(*)$. On the other hand, if for instance one considers annular domains

$$\Omega_{r_1,r_2} = \{x \in \mathbb{R}^n : 0 < r_1 < |x| < r_2\},$$

then the radial embedding

$$W_{0,rad}^{1,p}(\Omega_{r_1,r_2}) \hookrightarrow L^q(\Omega_{r_1,r_2})$$

Both authors: Dip. di Matem. e Fisica, Via Musei 41, I–25121 Brescia, Italy
squassin@dmf.unicatt.it and http://www.dmf.unicatt.it/∼squassin
musesti@dmf.unicatt.it and http://www.dmf.unicatt.it/∼musesti

ISSN 0232-2064 / \$ 2.50

is compact for each $q < +\infty$ and one can find a non-trivial radial solution of problem $(*)$ (see [11]). In particular, the existence of non-trivial solutions of problem $(*)$ depends also on the topology of the domain. In the case $p = 2$, the problem

$$\left.\begin{aligned} -\Delta u &= u^{(n+2)/(n-2)} && \text{in } \Omega \\ u &> 0 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \qquad (**)$$

has been deeply studied and existence results have been obtained provided that Ω satisfies suitable assumptions. In the striking paper [3], Bahri and Coron have proved that if Ω has a non-trivial topology, i.e. if Ω has a non-trivial homology in some positive dimension, then problem $(**)$ always admits a non-trivial solution.

On the other hand, Dancer [8] constructed for each $n \geq 3$ a contractible domain Ω_n, homeomorphic to a ball, for which problem $(**)$ has a non-trivial solution. Therefore, we see how the existence of non-trivial solutions of problem $(**)$ is related to the shape of the domain and not just to the topology. See also [15] and references therein for more recent existence and multiplicity results.

We remark that, to the authors' knowledge, this kind of achievements are not known when $p \neq 2$. In our opinion, one of the main difficulties is the fact that, differently from the case $p = 2$, it is not proven that all positive smooth solutions of the equation $-\Delta_p u = u^{p^*-1}$ in $\mathbb{R}^n$ are Talenti's radial functions, which attain the best Sobolev constant (see Proposition 3.1).

Now, there is a second approach in the study of problem $(*)$, which in general does not require any geometrical or topological assumption on Ω, namely to investigate the asymptotic behaviour of solutions u_ε of problems with nearly critical growth

$$\left.\begin{aligned} -\Delta_p u &= |u|^{p^*-2-\varepsilon} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \qquad (***)$$

as $\varepsilon \to 0$. If Ω is a ball and $p = 2$, Atkinson and Peletier [2] showed in 1987 the blow-up of a sequence of radial solutions. The extension to the case $p \neq 2$ was achieved by Knaap and Peletier [12] in 1989. On a general bounded domain, instead, the study of limits of solutions of problem $(***)$ was performed by Garcia Azorero and Peral Alonso [9] around 1992.

Let now $\varepsilon > 0$ and consider the general class of Euler-Lagrange equations with nearly critical growth

$$\left.\begin{aligned} -\mathrm{div}\,(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s \mathcal{L}(x, u, \nabla u) &= |u|^{p^*-2-\varepsilon} u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega \end{aligned}\right\} \qquad (\mathcal{P}_\varepsilon)$$

associated with the functional $f_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f_\varepsilon(u) = \int_\Omega \mathcal{L}(x, u, \nabla u)\, dx - \tfrac{1}{p^*-\varepsilon} \int_\Omega |u|^{p^*-\varepsilon} dx. \qquad (1)$$

As noted in [18], in general these functionals are not even locally Lipschitz under natural growth assumptions. Nevertheless, via techniques of non-smooth critical point theory (see [18] and references therein) it can be shown that for each $\varepsilon > 0$ problem $(\mathcal{P}_\varepsilon)$ admits a non-trivial solution $u_\varepsilon \in W_0^{1,p}(\Omega)$.

Let u_ε be a solution of problem $(\mathcal{P}_\varepsilon)$. The main goal of this paper is to prove that if the weak limit of $(|\nabla u_\varepsilon|^p)_{\varepsilon>0}$ has no blow-up points in Ω, then the limit problem

$$\left.\begin{aligned} -\mathrm{div}\,(\nabla_\xi \mathcal{L}(x,u,\nabla u)) + D_s\mathcal{L}(x,u,\nabla u) = |u|^{p^*-2}u \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned}\right\} \qquad (\mathcal{P}_0)$$

has a non-trivial solution (the weak limit of $(u_\varepsilon)_{\varepsilon>0}$), provided that $f_\varepsilon(u_\varepsilon) \to c$ with

$$\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} \tag{2}$$

where $\nu > 0$ and $\gamma \in (0, p^* - p)$ will be introduced later on. In our framework, (2) plays the role of a generalized second critical energy range (if $\gamma = 0$ and $\nu = 1$, one finds the usual range $\frac{S^{n/p}}{n} < c < 2\frac{S^{n/p}}{n}$ for problem $(***)$).

The plan of the paper is as follows:

In Section 2 we shall state our main results. Section 3 contains some preliminary lemmas, namely the lower bounds of the non-vanishing Dirac masses and of the non-trivial weak limits. In Section 4 we prove our main results. In Section 5 we see that at the mountain pass levels the sequence $(u_\varepsilon)_{\varepsilon>0}$ blows up. Finally, Section 6 contains a non-existence result.

2. The main results

Let Ω be any bounded domain of $\mathbb{R}^n$. In the following, the space $W_0^{1,p}(\Omega)$ will be endowed with the standard norm $\|u\|_{1,p}^p = \int_\Omega |\nabla u|^p dx$ and $\|\cdot\|_p$ will denote the usual norm of $L^p(\Omega)$.

Assume that $\mathcal{L} : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class C^1 in (s,ξ) a.e. in Ω, that $\mathcal{L}(x,s,\cdot)$ is strictly convex and $\mathcal{L}(x,s,0) = 0$. Moreover, assume the following:

$(\mathcal{A}_1)$ There exists $b_0 > 0$ such that

$$\mathcal{L}(x,s,\xi) \le b_0|s|^p + b_0|\xi|^p \tag{3}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

$(\mathcal{A}_2)$ There exists $b_1 > 0$ such that for each $\delta > 0$ there exists $a_\delta \in L^1(\Omega)$ with

$$|D_s\mathcal{L}(x,s,\xi)| \le a_\delta(x) + \delta|s|^{p^*} + b_1|\xi|^p \tag{4}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, there exist $a_1 \in L^{p'}(\Omega)$ and $\nu > 0$ such that

$$|\nabla_\xi \mathcal{L}(x,s,\xi)| \le a_1(x) + b_1|s|^{\frac{p^*}{p'}} + b_1|\xi|^{p-1}, \tag{5}$$

$$\nabla_\xi \mathcal{L}(x,s,\xi)\cdot \xi \geq \nu|\xi|^p \tag{6}$$

for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$.

$(\mathcal{A}_3)$ For a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$,

$$D_s\mathcal{L}(x,s,\xi)s\geq 0 \tag{7}$$

and there exists $\gamma\in(0,p^*-p)$ such that

$$(\gamma+p)\mathcal{L}(x,s,\xi)-\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\xi-D_s\mathcal{L}(x,s,\xi)s\geq 0 \tag{8}$$

for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$.

Remark 2.1. The growth conditions of $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ and the assumptions in $(\mathcal{A}_3)$ are natural in the fully nonlinear setting and were considered in [18], and in a stronger form in [1, 16] (see also Remark 6.2). Notice that when $\mathcal{L}$ is $p-$homogeneous with respect to ξ, then condition (8) becomes $D_s\mathcal{L}(x,s,\xi)s\leq\gamma\mathcal{L}(x,s,\xi)$ for a.e. $x\in\Omega$ and for all $(s,\xi)\in\mathbb{R}\times\mathbb{R}^n$.

As an example, taking $A\in C^1(\mathbb{R})\cap L^\infty(\mathbb{R})$ with $A'\in L^\infty(\mathbb{R})$, $A(s)\geq\nu$ and $\gamma A(s)\geq A'(s)s\geq 0$ for each $s\in\mathbb{R}$, the class of Lagrangians

$$\mathcal{L}(x,s,\xi)=\frac{1}{p}A(s)|\xi|^p$$

satisfies all the previous requirements. For instance $(\gamma^{-1}+\arctan(s^2))|\xi|^p/p$ belongs to this class for each $\gamma\in(0,p^*-p)$.

Remark 2.2. We stress that although as noted in the introduction f_ε fails to be differentiable, one may compute the derivatives along the $L^\infty-$directions, namely

$$f'_\varepsilon(u)(\varphi)=\int_\Omega\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\varphi\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)\varphi\,dx-\int_\Omega|u|^{p^*-2-\varepsilon}u\varphi\,dx.$$

for all $u\in W^{1,p}_0(\Omega)$ and for all $\varphi\in W^{1,p}_0\cap L^\infty(\Omega)$.

The following is a general property due to Brézis and Browder [5].

Proposition 2.3. *Let $u,v\in W^{1,p}_0(\Omega)$ be such that $D_s\mathcal{L}(x,u,\nabla u)v\geq 0$ and*

$$\langle w,\varphi\rangle=\int_\Omega\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\varphi\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)\varphi\,dx\qquad(\varphi\in C^\infty_c(\Omega))$$

with $w\in L^1_{loc}(\Omega)\cap W^{-1,p'}(\Omega)$. Then $D_s\mathcal{L}(x,u,\nabla u)v\in L^1(\Omega)$ and

$$\langle w,v\rangle=\int_\Omega\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla v\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)v\,dx.$$

From now on, by solution of problem $(\mathcal{P}_\varepsilon)$ we shall always mean weak solution, namely $f'_\varepsilon(u_\varepsilon)=0$ in the sense of distributions. The next lemma is our starting point.

Lemma 2.4. *For each $\varepsilon > 0$, $(\mathcal{P}_\varepsilon)$ admits a non-trivial solution $u_\varepsilon \in W_0^{1,p}(\Omega)$.*

Proof. See [18: Theorem 1.1] ∎

We point out that, in our general framework, the technical aspects in the verification of the Palais-Smale condition for f_ε are, in our opinion, interesting and not trivial.

Note that since $\mathcal{L}(x,s,0) = 0$, in view of (6) one obtains

$$\mathcal{L}(x,s,\xi) \geq \tfrac{\nu}{p}\,|\xi|^p \tag{9}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Lemma 2.5. *Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ such that $\lim\limits_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) < +\infty$. Then $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.*

Proof. For each $\varepsilon > 0$ we have $f'_\varepsilon(u_\varepsilon)(\varphi) = 0$ for each $\varphi \in C_c^\infty(\Omega)$. On the other hand, taking into account (7), by Proposition 2.3 one can also take $\varphi = u_\varepsilon$. Therefore, in view of (8) and (9) one obtains

$$\begin{aligned}
\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) &= \lim_{\varepsilon\to 0}\Big(f_\varepsilon(u_\varepsilon) - \tfrac{1}{p^*-\varepsilon} f'_\varepsilon(u_\varepsilon)(u_\varepsilon)\Big)\\
&= \lim_{\varepsilon\to 0}\bigg(\int_\Omega \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx\\
&\qquad - \tfrac{1}{p^*-\varepsilon}\int_\Omega \nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon\,dx\\
&\qquad - \tfrac{1}{p^*-\varepsilon}\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)u_\varepsilon dx\bigg)\\
&\geq \lim_{\varepsilon\to 0}\tfrac{p^*-p-\varepsilon-\gamma}{p^*-\varepsilon}\int_\Omega \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx\\
&\geq \tfrac{p^*-p-\gamma}{pp^*}\nu\lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p dx.
\end{aligned}$$

In particular, $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$ ∎

Let us now recall the classical P.L. Lions' concentration-compactness principle

Lemma 2.6. *Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be bounded and let u be its weak limit. Then there exist two bounded positive measures μ and σ such that*

$$|\nabla u_\varepsilon|^p \rightharpoonup \mu,\ \ |u_\varepsilon|^{p^*} \rightharpoonup \sigma \quad (\textit{in the sense of measures}) \tag{10}$$

$$\mu \geq |\nabla u|^p + \sum_{j=1}^\infty \mu_j\delta_{x_j} \quad (\mu_j \geq 0) \tag{11}$$

$$\sigma = |u|^{p^*} + \sum_{j=1}^\infty \sigma_j\delta_{x_j} \quad (\sigma_j \geq 0) \tag{12}$$

$$\mu_j \geq S\sigma_j^{\frac{p}{p^*}} \tag{13}$$

where δ_{x_j} denotes the Dirac measure at $x_j \in \overline{\Omega}$ and S denotes the best Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$.

Proof. See e.g. [13, Lemma I.1] or [14] ∎

Under assumptions $(\mathcal{A}_1)-(\mathcal{A}_3)$, the following is our main result.

Theorem 2.7. *Assume that $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ is a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ such that $f_\varepsilon(u_\varepsilon) \to c$ and*

$$\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}.$$

Then $\mu_j = 0$ for $j \geq 2$ and the following alternative holds:

(a) *$\mu_1 = 0$ and u is a non-trivial solution of problem $(\mathcal{P}_0)$.*

(b) *$\mu_1 \neq 0$ and $u = 0$.*

This result extends [9: Theorem 9] to a class of fully nonlinear elliptic problems.

Theorem 2.8. *Let $(u_\varepsilon)_{\varepsilon>0}$ be any sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ with*

$$\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) = \tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}.$$

Then $u = 0$.

As we shall see in Section 5, this is also the behaviour when one considers critical levels of mountain-pass type.

3. The weak limit of $(u_\varepsilon)_{\varepsilon>0}$

Let us briefly summarize the main properties of the best Sobolev constant [19].

Proposition 3.1. *Let $1 < p < n$ and S be the best Sobolev constant, i.e.*

$$S = \inf\left\{\int_\Omega |\nabla u|^p dx : \, u \in W_0^{1,p}(\Omega) \quad \text{with} \quad \int_\Omega |u|^{p^*} dx = 1\right\}. \tag{14}$$

Then the following facts hold:

(a) *S is independent on $\Omega \subset \mathbb{R}^n$.*

(b) *The infimum (14) is never achieved on bounded domains $\Omega \subset \mathbb{R}^n$.*

(c) *The infimum (14) is achieved if $\Omega = \mathbb{R}^n$ by the family of functions on $\mathbb{R}^n$*

$$T_{\delta,x_0}(x) = \left(n\delta\left(\tfrac{n-p}{p-1}\right)^{p-1}\right)^{\frac{n-p}{p^2}} \left(\delta + |x-x_0|^{\frac{p}{p-1}}\right)^{-\frac{n-p}{p}} \tag{15}$$

with $\delta > 0$ and $x_0 \in \mathbb{R}^n$. Moreover, T_{δ,x_0} is a solution of $-\Delta_p u = u^{p^*-1}$ on $\mathbb{R}^n$.

The next result establishes uniform lower bounds for the Dirac masses.

Lemma 3.2. *If $\mu_j \neq 0$, then $\sigma_j \geq \nu^{\frac{n}{p}} S^{\frac{n}{p}}$ and $\mu_j \geq \nu^{\frac{n}{p^*}} S^{\frac{n}{p}}$.*

Proof. Let $x_j \in \overline{\Omega}$ the point which supports the Dirac measure of coefficient σ_j. Denoting with $B(x_j, \delta)$ the open ball of center x_j and radius $\delta > 0$, we can consider a function $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \psi_\delta \leq 1$, $|\nabla\psi_\delta| \leq \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$. By Proposition 2.3 we have

$$
\begin{aligned}
0 &= f'_\varepsilon(u_\varepsilon)(\psi_\delta u_\varepsilon) \\
&= \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\psi_\delta \, dx + \int_\Omega \psi_\delta \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon dx \\
&\quad + \int_\Omega \psi_\delta D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon dx - \int_\Omega |u_\varepsilon|^{p^*-\varepsilon} \psi_\delta dx.
\end{aligned} \tag{16}
$$

Applying Hölder inequality and (5) to the first term of the decomposition and keeping into account that $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$, one finds constants $c_1, c_2 > 0$ such that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \left| \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\psi_\delta dx \right| \\
&\leq \left(\int_{B(x_j,2\delta)} |a_1|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left(\int_{B(x_j,2\delta)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_{B(x_j,2\delta)} |\nabla\psi_\delta|^n dx \right)^{\frac{1}{n}} \\
&\quad + b_1 \left(\int_{B(x_j,2\delta)} |u|^{p^*} dx \right)^{\frac{n-1}{n}} \left(\int_{B(x_j,2\delta)} |\nabla\psi_\delta|^n \, dx \right)^{\frac{1}{n}} \\
&\quad + \widetilde{b}_1 \left(\int_{B(x_j,2\delta)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_{B(x_j,2\delta)} |\nabla\psi_\delta|^n dx \right)^{\frac{1}{n}} \\
&\leq c_1 \left(\int_{B(x_j,2\delta)} |u|^{p^*} dx \right)^{\frac{1}{p^*}} + c_2 \left(\int_{B(x_j,2\delta)} |u|^{p^*} dx \right)^{\frac{n-1}{n}} = \beta_\delta
\end{aligned} \tag{17}
$$

with $\beta_\delta \to 0$ as $\delta \to 0$. Then, taking into account (6) and (7) one has

$$
\begin{aligned}
0 &\geq -\beta_\delta + \lim_{\varepsilon\to 0} \nu \int_\Omega |\nabla u_\varepsilon|^p \psi_\delta dx - \lim_{\varepsilon\to 0} \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}} \left(\int_\Omega |u_\varepsilon|^{p^*} \psi_\delta dx \right)^{\frac{p^*-\varepsilon}{p^*}} \\
&\geq -\beta_\delta + \nu \int_\Omega \psi_\delta d\mu - \int_\Omega \psi_\delta d\sigma.
\end{aligned}
$$

Letting $\delta \to 0$, it results $\nu\mu_j \leq \sigma_j$. By means of (13) the proof is complete ∎

In the next result we obtain uniform lower bounds for the non-zero weak limits.

Lemma 3.3. *If $u \neq 0$, then $\int_\Omega |\nabla u|^p dx > \nu^{\frac{n}{p^*}} S^{\frac{n}{p}}$ and $\int_\Omega |u|^{p^*} dx > \nu^{\frac{n}{p}} S^{\frac{n}{p}}$.*

Proof. By Lemma 3.2 we may assume that μ has at most r Dirac masses $\mu_1, \ldots, \mu_r$ at $x_1, \ldots, x_r$, respectively. Let now $0 < \delta < \frac{1}{4} \min\{|x_i - x_j| : i \neq j\}$

and $\psi_\delta \in C_c^\infty(\mathbb{R}^n)$ be such that $0 \le \psi_\delta \le 1$, $|\nabla \psi_\delta| \le \frac{2}{\delta}$, $\psi_\delta(x) = 1$ if $x \in B(x_j, \delta)$ and $\psi_\delta(x) = 0$ if $x \notin B(x_j, 2\delta)$. Taking into account (7), for each $\varepsilon, \delta > 0$ we have

$$\int_\Omega D_s\mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta)\, dx \ge 0.$$

Then, since one can choose $(1 - \psi_\delta) u_\varepsilon$ as test, by (6) one obtains

$$\begin{aligned} 0 &= f_\varepsilon'(u_\varepsilon)((1 - \psi_\delta) u_\varepsilon) \\ &= \int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon (1 - \psi_\delta)\, dx \\ &\quad - \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta dx \\ &\quad + \int_\Omega D_s \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) u_\varepsilon (1 - \psi_\delta)\, dx \\ &\quad - \int_\Omega |u_\varepsilon|^{p^* - \varepsilon} (1 - \psi_\delta)\, dx \\ &\ge \nu \int_\Omega |\nabla u_\varepsilon|^p (1 - \psi_\delta)\, dx \\ &\quad - \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta dx \\ &\quad - \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}} \left(\int_\Omega |u_\varepsilon|^{p^*} (1 - \psi_\delta)\, dx \right)^{\frac{p^* - \varepsilon}{p^*}}. \end{aligned} \tag{18}$$

On the other hand, arguing as for (17), one obtains

$$\lim_{\varepsilon \to 0} \left| \int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla \psi_\delta dx \right| \le \beta_\delta \tag{19}$$

for each $\delta > 0$. Now, it results

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega |\nabla u_\varepsilon|^p (1 - \psi_\delta)\, dx &= \int_\Omega (1 - \psi_\delta)\, d\mu \\ &\ge \int_\Omega |\nabla u|^p (1 - \psi_\delta)\, dx + \sum_{j=1}^r \mu_j (1 - \psi_\delta(x_j)) \\ &= \int_\Omega |\nabla u|^p dx + o(1) \end{aligned} \tag{20}$$

as $\delta \to 0$ and

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_\Omega |u_\varepsilon|^{p^*} (1 - \psi_\delta)\, dx &= \int_\Omega (1 - \psi_\delta)\, d\sigma \\ &= \int_\Omega |u|^{p^*} (1 - \psi_\delta)\, dx + \sum_{j=1}^r \sigma_j (1 - \psi_\delta(x_j)) \\ &= \int_\Omega |u|^{p^*} dx + o(1) \end{aligned} \tag{21}$$

as $\delta \to 0$. Therefore, in view of (19) - (21), by letting $\delta \to 0$ and $\varepsilon \to 0$ in (18) one concludes that

$$\nu \int_\Omega |\nabla u|^p dx \le \int_\Omega |u|^{p^*} dx. \tag{22}$$

As Ω is bounded, by Proposition 3.1/(b) one has $\int_\Omega |\nabla u|^p dx > S \left(\int_\Omega |u|^{p^*} dx\right)^{\frac{p}{p^*}}$ which combined with (22) yields the assertion ∎

Lemma 3.4. *Let $(u_\varepsilon)_{\varepsilon>0} \subset W_0^{1,p}(\Omega)$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ and let u be its weak limit. Then u is a solution of problem $(\mathcal{P}_0)$.*

Proof. For each $\varepsilon > 0$ and $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\varphi \, dx + \int_\Omega D_s\mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon)\varphi \, dx = \int_\Omega |u_\varepsilon|^{p^*-2-\varepsilon} u_\varepsilon \varphi \, dx. \tag{23}$$

Since $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence, u satisfies

$$\left.\begin{array}{rl} \nabla u_\varepsilon \rightharpoonup \nabla u & \text{in } L^p(\Omega) \\ u_\varepsilon \to u & \text{in } L^p(\Omega) \\ u_\varepsilon(x) \to u(x) & \text{for a.e. } x \in \Omega \end{array}\right\}$$

as $\varepsilon \to 0$. Moreover, by [7: Theorem 1], up to a further subsequence, we have $\nabla u_\varepsilon(x) \to \nabla u(x)$ for a.e. $x \in \Omega$. Therefore, in view of (5) one deduces that

$$\nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \rightharpoonup \nabla_\xi \mathcal{L}(x, u, \nabla u) \qquad \text{in } L^{p'}(\Omega, \mathbb{R}^n). \tag{24}$$

By (4) - (6) one finds a constant $M > 0$ such that for each $\delta > 0$

$$|D_s\mathcal{L}(x, s, \xi)| \le M \nabla_\xi \mathcal{L}(x, s, \xi) \cdot \xi + a_\delta(x) + \delta |s|^{p^*} \tag{25}$$

for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. If we test equation (23) with the functions

$$\varphi_\varepsilon = \varphi \exp\{-M u_\varepsilon^+\} \qquad (\varepsilon > 0)$$

where $0 \le \varphi \in W_0^{1,p} \cap L^\infty(\Omega)$, we obtain

$$\begin{aligned} &\int_\Omega \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla\varphi \exp\{-M u_\varepsilon^+\} dx \\ &\quad - \int_\Omega |u_\varepsilon|^{p^*-2-\varepsilon} u_\varepsilon \varphi \exp\{-M u_\varepsilon^+\} dx \\ &+ \int_\Omega \Big[D_s\mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - M \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^+ \Big] \varphi \exp\{-M u_\varepsilon^+\} dx = 0. \end{aligned}$$

Since by inequalities (7) and (25) for each $\varepsilon > 0$ and $\delta > 0$ we have

$$\Big[D_s\mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) - M \nabla_\xi \mathcal{L}(x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla u_\varepsilon^+ \Big] \varphi \exp\{-M u_\varepsilon^+\} - \delta |u_\varepsilon|^{p^*} \le a_\delta(x),$$

arguing as in [18: Theorem 3.4] one obtains

$$\limsup_{\varepsilon\to 0}\int_\Omega \Big[D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)-M\nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon^+\Big]\varphi\exp\{-Mu_\varepsilon^+\}dx$$
$$\leq \int_\Omega \Big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u^+\Big]\varphi\exp\{-Mu^+\}dx.$$

Therefore, taking into account (24) and since as $\varepsilon\to 0$

$$\int_\Omega |u_\varepsilon|^{p^*-2-\varepsilon}u_\varepsilon\varphi\,dx\to\int_\Omega |u|^{p^*-2}u\varphi\,dx$$

for each $0\leq\varphi\in W_0^{1,p}\cap L^\infty(\Omega)$, one may conclude that

$$\begin{aligned}\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\varphi\exp\{-Mu^+\}dx&\\ -\int_\Omega |u|^{p^*-2}u\varphi\exp\{-Mu^+\}dx&\\ +\int_\Omega \Big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u^+\Big]\varphi\exp\{-Mu^+\}dx\geq 0&\end{aligned}\tag{26}$$

for each $0\leq\varphi\in W_0^{1,p}\cap L^\infty(\Omega)$. Testing now (26) with $\varphi_k=\varphi\vartheta\left(\frac{u}{k}\right)\exp\{Mu^+\}$ where $0\leq\varphi\in C_c^\infty(\Omega)$ and ϑ is smooth, $\vartheta=1$ in $\left[-\frac{1}{2},\frac{1}{2}\right]$ and $\vartheta=0$ in $(-\infty,-1]\cup[1,+\infty)$, it follows that

$$\begin{aligned}\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\varphi_k\exp\{-Mu^+\}dx&\\ -\int_\Omega |u|^{p^*-2}u\varphi\,\vartheta\left(\tfrac{u}{k}\right)dx&\\ +\int_\Omega \Big[D_s\mathcal{L}(x,u,\nabla u)-M\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u^+\Big]\varphi\,\vartheta\left(\tfrac{u}{k}\right)dx\geq 0&\end{aligned}$$

which, arguing again as in [18: Theorem 3.4], yields as $k\to+\infty$

$$\int_\Omega \nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\varphi\,dx+\int_\Omega D_s\mathcal{L}(x,u,\nabla u)\varphi\,dx\geq\int_\Omega |u|^{p^*-2}u\varphi\,dx$$

for each $0\leq\varphi\in C_c^\infty(\Omega)$. Analogously, testing with $\varphi_\varepsilon=\varphi\exp\{-Mu_\varepsilon^-\}$, one obtains the opposite inequality, i.e. u is a solution of problem $(\mathcal{P}_0)$ ∎

4. Proofs of the main results

Let now $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ with $f_\varepsilon(u_\varepsilon) \to c$ and

$$\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < 2\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}. \tag{27}$$

Then there exist a subsequence of $(u_\varepsilon)_{\varepsilon>0}$ and two bounded positive measures μ and σ verifying (10) - (13).

Proof of Theorem 2.7. Let us first show that there exists at most one j such that $\mu_j \neq 0$. Suppose that $\mu_j \neq 0$ for $j = 1, \dots, r$; in view of Lemma 3.2 one has $\mu_j \geq \nu^{\frac{n}{p^*}} S^{\frac{n}{p}}$. Following the proof of Lemma 2.5, we obtain

$$\begin{aligned} c &= \lim_{\varepsilon \to 0} f_\varepsilon(u_\varepsilon) \\ &\geq \frac{p^*-p-\gamma}{pp^*}\nu \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p dx \\ &\geq \frac{p^*-p-\gamma}{pp^*}\nu \int_\Omega d\mu \\ &\geq \frac{p^*-p-\gamma}{pp^*}\nu \sum_{j=1}^r \mu_j \\ &\geq r\,\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}. \end{aligned}$$

Taking into account (27) one has

$$2\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} > c \geq r\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}},$$

hence $r \leq 1$. Now, arguing again as in Lemma 2.5 one obtains

$$\begin{aligned} 2\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} > c &= \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\nu \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^p dx \\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\left(\nu\int_\Omega |\nabla u|^p dx + \nu\mu_1\right). \end{aligned}$$

If both summands were non-zero, by Lemmas 3.2 and 3.3 we would obtain

$$\begin{aligned} \nu\int_\Omega |\nabla u|^p dx &> (\nu S)^{\frac{n}{p}}, \\ \nu\mu_1 &\geq (\nu S)^{\frac{n}{p}} \end{aligned}$$

and thus a contradiction. Vice versa, let us assume that $u = 0$ and $\mu_1 = 0$. Let $0 \leq \psi \in C_c^1(\Omega)$. By testing our equation with ψu_ε and using Hölder inequality, one gets

$$\begin{aligned}
&\int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi\,dx \\
&+\int_\Omega \psi\nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx \\
&\quad+\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\psi u_\varepsilon dx = \int_\Omega |u_\varepsilon|^{p^*-\varepsilon}\psi dx \\
&\qquad\qquad\qquad\qquad \le \left(\int_\Omega |u_\varepsilon|^{p^*}\psi\,dx\right)^{\frac{p^*-\varepsilon}{p^*}} \mathcal{L}^n(\Omega)^{\frac{\varepsilon}{p^*}}
\end{aligned} \tag{28}$$

Since $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$, by (5) there exists a constant $C>0$ such that

$$\left|\int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi\,dx\right| \le C\,\|u_\varepsilon\|_p$$

which by $u_\varepsilon \to 0$ in $L^p(\Omega)$ yields

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon \nabla_\xi \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla\psi\,dx = 0.$$

Moreover, since by (7) we get

$$\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\psi u_\varepsilon dx \ge 0,$$

taking into account (6) and passing to the limit in (28) we get

$$\forall\,\psi\in C_c(\Omega):\quad \psi\ge 0 \quad\Longrightarrow\quad \nu\int_\Omega \psi\,d\mu \le \int_\Omega \psi\,d\sigma. \tag{29}$$

On the other hand, $\mu_1=0$ and $u=0$ imply $\sigma=0$. Then, since $\mu\ge 0$, by (29) we get $\mu=0$. In particular, by (3), (6) and (7) one gets

$$\begin{aligned}
c &= \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \\
&= \lim_{\varepsilon\to 0}\Bigg[\int_\Omega \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx \\
&\quad - \tfrac{1}{p^*-\varepsilon}\int_\Omega \nabla_\xi\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx \\
&\quad - \tfrac{1}{p^*-\varepsilon}\int_\Omega D_s\mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon) u_\varepsilon dx\Bigg] \\
&\le b_0 \lim_{\varepsilon\to 0}\left(\int_\Omega |u_\varepsilon|^p dx + \int_\Omega |\nabla u_\varepsilon|^p dx\right) \\
&= b_0\int_\Omega d\mu \\
&= 0,
\end{aligned}$$

which is not possible. Therefore, either $\mu_1=0$ and $u\ne 0$, or $\mu_1\ne 0$ and $u=0$ ∎

Remark 4.1. If (27) is replaced by the $(k+1)$-th critical energy range

$$k\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < c < (k+1)\,\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}$$

for $k \in \mathbb{N}$, in a similar way one proves that $\mu_j = 0$ for any $j \geq k+1$ and there holds:

(a) If $\mu_j = 0$ for every $j \geq 1$, then u is a non-trivial solution of problem $(\mathcal{P}_0)$.

(b) If $\mu_j \neq 0$ for every $1 \leq j \leq k$, then $u = 0$.

Remark 4.2. Let $f_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the functional associated with problem $(\mathcal{P}_0)$ and let $0 \neq u \in W_0^{1,p}(\Omega)$ be a solution of problem $(\mathcal{P}_0)$ (obtained as weak limit of $(u_\varepsilon)_{\varepsilon>0}$). Then

$$f_0(u) > \tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}. \tag{30}$$

Indeed,

$$\begin{aligned} f_0(u) &= f_0(u) - \tfrac{1}{p^*} f_0'(u)(u) \\ &\geq \tfrac{p^*-p-\gamma}{p^*} \int_\Omega \mathcal{L}(x,u,\nabla u)\,dx \\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\nu \int_\Omega |\nabla u|^p dx \end{aligned}$$

which yields (30) in view of Lemma 3.3. This, in some sense, explains why one chooses c greater than $\frac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}$ in Theorem 2.7.

Let now $(u_\varepsilon)_{\varepsilon>0}$ be a sequence of solutions of problem $(\mathcal{P}_\varepsilon)$ with $f_\varepsilon(u_\varepsilon) \to c$ and

$$\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) = \tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}.$$

Proof of Theorem 2.8. Let us first note that

$$f_0(u) \leq \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) + \tfrac{1}{p^*}\sum_{j=1}^{\infty}\sigma_j. \tag{31}$$

Indeed, taking into account that by [6: Theorem 3.4]

$$\int_\Omega \mathcal{L}(x,u,\nabla u)\,dx \leq \lim_{\varepsilon\to 0}\int_\Omega \mathcal{L}(x,u_\varepsilon,\nabla u_\varepsilon)\,dx,$$

(31) follows by combining Hölder inequality with (12).

Now assume by contradiction that $u \neq 0$. Then there exists $j_0 \in \mathbb{N}$ such that $\mu_{j_0} \neq 0$ and $\sigma_{j_0} \neq 0$, otherwise by Remark 4.2 and (31) we would get

$$\tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} < f_0(u) \leq \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) = \tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}}.$$

Arguing as in Lemma 2.5 and applying Lemma 3.2, we obtain

$$\begin{aligned} \tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} &= \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\left(\nu\int_\Omega |\nabla u|^p dx + \nu\mu_{j_0}\right) \\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\nu\int_\Omega |\nabla u|^p dx + \tfrac{p^*-p-\gamma}{pp^*}(\nu S)^{\frac{n}{p}} \end{aligned}$$

which implies $u = 0$ – a contradiction ∎

5. Mountain-pass critical values

In this section, we shall investigate the asymptotics of (u_ε) in the case of critical levels of min-max type. We assume that $\mathcal{L}$ is p-homogeneous with respect to ξ and satisfies a stronger assumption, i.e.

$$\mathcal{L}(x,s,\xi) \le \tfrac{1}{p}\,|\xi|^p \tag{32}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$. In particular, it results that $\nu \le 1$. Let u_ε be a critical point of f_ε associated with the mountain pass level

$$c_\varepsilon = \inf_{\eta\in\mathcal{C}_\varepsilon}\max_{t\in[0,1]} f_\varepsilon(\eta(t)) \tag{33}$$

where

$$\mathcal{C}_\varepsilon = \Big\{\eta \in C\big([0,1], W_0^{1,p}(\Omega)\big) : \eta(0)=0 \text{ and } \eta(1)=w_\varepsilon\Big\}$$

and $w_\varepsilon \in W_0^{1,p}(\Omega)$ is chosen in such a way that $f_\varepsilon(w_\varepsilon) < 0$.

Lemma 5.1. *The inequality* $\lim\limits_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \le \frac{1}{n}S^{\frac{n}{p}}$ *holds.*

Proof. Let $x_0 \in \Omega$ and $\delta > 0$, and consider the functions T_{δ,x_0} as in (15). By Proposition 3.1/(c) one has

$$\|\nabla T_{\delta,x_0}\|^p_{p,\mathbb{R}^n} = \|T_{\delta,x_0}\|^{p^*}_{p^*,\mathbb{R}^n} = S^{\frac{n}{p}}.$$

Moreover, taking a function $\phi \in C_c^\infty(\Omega)$ with $0 \le \phi \le 1$ and $\phi = 1$ in a neighbourhood of x_0 and setting $v_\delta = \phi T_{\delta,x_0}$, it results

$$\left.\begin{aligned} \|\nabla v_\delta\|_p^p &= S^{\frac{n}{p}} + o(1) \\ \|v_\delta\|_{p^*}^{p^*} &= S^{\frac{n}{p}} + o(1) \end{aligned}\right\} \qquad (\delta \to 0) \tag{34}$$

(see [10: Lemma 3.2]).

We want to prove that, for any $t \ge 0$,

$$\lim_{\varepsilon\to 0} f_\varepsilon(tv_\delta) \le \tfrac{1}{n}S^{\frac{n}{p}} + o(1) \qquad (\delta\to 0).$$

By (32) one has

$$\begin{aligned} \lim_{\varepsilon\to 0} f_\varepsilon(tv_\delta) &= t^p\int_\Omega \mathcal{L}(x,tv_\delta,\nabla v_\delta)\,dx - \lim_{\varepsilon\to 0}\tfrac{t^{p^*-\varepsilon}}{p^*-\varepsilon}\int_\Omega |v_\delta|^{p^*-\varepsilon}dx \\ &\le \tfrac{t^p}{p}\int_\Omega |\nabla v_\delta|^p dx - \tfrac{t^{p^*}}{p^*}\int_\Omega |v_\delta|^{p^*}dx. \end{aligned}$$

Keeping into account (34) and the fact that $\frac{t^p}{p} - \frac{t^{p^*}}{p^*} \le \frac{1}{n}$ for every $t \ge 0$, one gets

$$\lim_{\varepsilon\to 0} f_\varepsilon(tv_\delta) \le \tfrac{t^p}{p}S^{\frac{n}{p}} - \tfrac{t^{p^*}}{p^*}S^{\frac{n}{p}} + o(1) \le \tfrac{1}{n}S^{\frac{n}{p}} + o(1) \qquad (\delta\to 0).$$

Now choose $t_0 > 0$ such that $f_\varepsilon(t_0v_\delta) < 0$; by (33) we have

$$\lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon) \le \lim_{\varepsilon\to 0}\max_{s\in[0,1]} f_\varepsilon(st_0v_\delta) \le \tfrac{1}{n}S^{\frac{n}{p}} + o(1)$$

and this, by letting $\delta \to 0$, ends up the proof ∎

Theorem 5.2. *Suppose that the number of non-zero Dirac masses is*

$$\left[\frac{pp^*}{(p^*-p-\gamma)n\nu^{\frac{n}{p}}}\right]$$

where $[x]$ *denotes the integer part of* x. *Then* $u = 0$.

Proof. Keeping into account the previous lemma and arguing as in Lemma 2.5,

$$\begin{aligned}\tfrac{1}{n}S^{\frac{n}{p}} &\geq \lim_{\varepsilon\to 0} f_\varepsilon(u_\varepsilon)\\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\nu\left(\int_\Omega |\nabla u|^p dx + \sum_{j=1}^r \mu_j\right)\\ &\geq \tfrac{p^*-p-\gamma}{pp^*}\nu\int_\Omega |\nabla u|^p dx + r\,\tfrac{p^*-p-\gamma}{pp^*}\nu^{\frac{n}{p}}S^{\frac{n}{p}}\end{aligned}$$

where r denotes the number of non-vanishing masses. Hence it must be

$$0 \leq r \leq \left[\frac{pp^*}{(p^*-p-\gamma)n\nu^{\frac{n}{p}}}\right].$$

In particular, if r is maximum and $u \neq 0$, by virtue of Lemma 3.3 one obtains

$$\tfrac{p^*-p-\gamma}{pp^*}\nu^{\frac{n}{p}}S^{\frac{n}{p}} > \tfrac{p^*-p-\gamma}{pp^*}\nu\int_\Omega |\nabla u|^p dx > \tfrac{p^*-p-\gamma}{pp^*}\nu^{\frac{n}{p}}S^{\frac{n}{p}}$$

which is a contradiction ∎

6. Final remarks

Assume that $\mathcal{L}(x,s,\xi)$ is of class C^1 in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$ and, additionally, that the vector-valued function

$$\nabla_\xi\mathcal{L}(x,s,\xi) = \left(\frac{\partial\mathcal{L}}{\partial\xi_1}(x,s,\xi),\ldots,\frac{\partial\mathcal{L}}{\partial\xi_n}(x,s,\xi)\right)$$

is of class C^1 in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$.

Theorem 6.1. *Let* Ω *be star-shaped with respect to the origin and assume that*

$$p^*\nabla_x\mathcal{L}(x,s,\xi)\cdot x - nD_s\mathcal{L}(x,s,\xi)s \geq 0$$

for $(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$. *Then* $(\mathcal{P}_0)$ *has no non-trivial solution* u *in* $C^2(\Omega)\cap C^1(\overline{\Omega})$.

Proof. Let $\widehat{\mathcal{L}}:\Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ by defined by setting

$$\widehat{\mathcal{L}}(x,s,\xi) = \mathcal{L}(x,s,\xi) - \tfrac{1}{p^*}|s|^{p^*}$$

for all $(x,s,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$. Then apply the Pucci-Serrin inequality [17]

$$n\widehat{\mathcal{L}} + \nabla_x\widehat{\mathcal{L}}\cdot x - aD_s\widehat{\mathcal{L}}s - (a+1)\nabla_\xi\widehat{\mathcal{L}}\cdot\xi \geq 0$$

with the choice $a = \frac{n-p}{p}$ ∎

Remark 6.2. If Ω is star-shaped and $\mathcal{L}$ does not depend on x, then problem $(\mathcal{P}_0)$ admits no non-trivial solution in $C^2(\Omega) \cap C^1(\overline{\Omega})$ when $D_s\mathcal{L}(s,\xi)s \leq 0$, which is the opposite of (7). In particular, (7) seems to be a natural assumption.

Remark 6.3. As noted in the introduction, if Ω is star-shaped and $\mathcal{L}(\xi) = |\xi|^p/p$, in [10] it is proven that problem $(\mathcal{P}_0)$ has no non-trivial solution in $W_0^{1,p}(\Omega)$. In particular, by Theorem 2.7 one has $\mu_1 \neq 0$.

Acknowledgement. The authors wish to thank M. Degiovanni for providing some useful discussions.

References

[1] Arcoya, D. and L. Boccardo: *Critical points for multiple integrals of the calculus of variations.* Arch. Ration. Mech. Anal. 134 (1996), 249 – 274.

[2] Atkinson, F. V. and L. A. Peletier: *Elliptic equations with nearly critical growth.* J. Diff. Equ. 70 (1987), 349 – 365.

[3] Bahri, A. and J. M. Coron: *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain.* Comm. Pure Appl. Math. 41 (1988), 253 – 294.

[4] Bandle, C. and A. Brillard: *Nonlinear elliptic equations involving critical Sobolev exponents: Asymptotic analysis via methods of epi-convergence.* Z. Anal. Anw. 13 (1994), 615 – 628.

[5] Brézis, H. and F. E. Browder: *Some properties of higher order Sobolev spaces.* J. Math. Pures Appl. 61 (1982), 245 – 259.

[6] Dacorogna, B.: *Direct Methods in the Calculus of Variations.* Berlin: Springer-Verlag 1988.

[7] Dal Maso, G. and F. Murat: *Almost everywhere convergence of the gradients of solutions to nonlinear elliptic systems.* Nonlin. Anal. 31 (1998), 405 – 412.

[8] Dancer, E. N.: *A note on an equation with critical exponent.* Bull. London Math. Soc. 20 (1988), 600 – 602.

[9] Garcia Azorero, J. and I. Peral Alonso: *On limits of solutions of elliptic problems with nearly critical exponent.* Comm. Part. Diff. Equ. 17 (1992), 2113 – 2126.

[10] Guedda, M. and L. Veron: *Quasilinear elliptic problems involving critical Sobolev exponents.* Nonlin. Anal. 13 (1989), 879 – 902.

[11] Kazdan, J. L. and F. W. Warner: *Remarks on some quasilinear elliptic equations.* Comm. Pure Appl. Math. 28 (1975), 567 – 597.

[12] Knaap, M. C. and L. A. Peletier: *Quasilinear elliptic equations with nearly critical growth.* Comm. Part. Diff. Equ. 14 (1989), 1351 – 1383.

[13] Lions, P. L.: *The concentration-compactness principle in the calculus of variations.* Part I: *The limit case.* Rev. Mat. Iberoamer. 1 (1985), (1) 145 – 201.

[14] Lions, P. L.: *The concentration-compactness principle in the calculus of variations.* Part II: *The limit case.* Rev. Mat. Iberoamer. 1 (1985), (2) 45 – 121.

[15] Passaseo, D.: *The effect of the domain shape on the existence of positive solutions of the equation* $\Delta u + u^{2^*-1} = 0$. Top. Meth. Nonlin. Anal. 3 (1994), 27 – 54.

[16] Pellacci, B.: *Critical points for non-differentiable functionals.* Bull. Un. Mat. Ital. B (7) 11 (1997), 733 – 749.

[17] Pucci, P. and J. Serrin: *A general variational identity.* Indiana Univ. Math. J. 35 (1986), 681 – 703.

[18] Squassina, M.: *Existence of weak solutions to general Euler's equations via nonsmooth critical point theory.* Ann. Fac. Sci. Toulouse Math. (6) 9 (2000), 113 – 131.

[19] Talenti, G.: *Best constant in Sobolev inequality.* Ann. Mat. Pura Appl. (4) 110 (1976), 353 – 372.

Received 07.06.2001