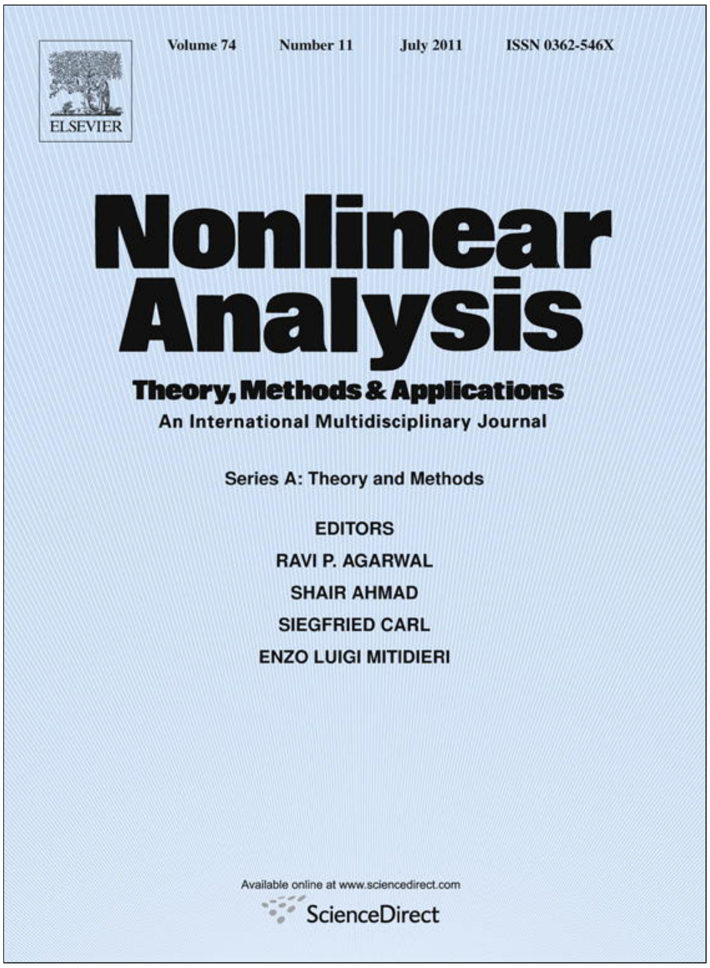
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On Palais' principle for non-smooth functionals

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ABSTRACT

If G is a compact Lie group acting linearly on a Banach space X and f is a G-invariant function on X, we provide some new versions of the so-called Palais' principle of symmetric criticality for $f : X \to \bar{\mathbb{R}}$, in the framework of non-smooth critical point theory. We apply the results to a class of quasi-linear PDEs associated with invariant functionals which are merely lower semi-continuous and thus could not be treated by previous non-smooth versions of the principle in the literature.

1. Introduction

Let G be a compact Lie group which acts linearly on a Banach space X, let $f : X \to \bar{\mathbb{R}}$ be a G-invariant functional and consider the fixed point set of X under G,

$$\mathrm{Fix}(G) = \bigcap_{g \in G} \{u \in X : (g - \mathrm{Id})u = 0\},$$

which is a closed subspace of X. In order to detect critical points of f, a possible approach is based upon what is currently known as Palais' principle of symmetric criticality, that is, looking for critical points of $f|_{\mathrm{Fix}(G)}$ in order to locate critical points of f. The origin of the principle is rather unclear and the first implicit use seems to trace back to 1950 inside Weyl derivation of the Einstein field equations [1]. Next, Coleman [2] made an explicit reference to it around 1975. For G-invariant functionals of class C^1 the principle was rigorously formulated by Palais in 1979 in a celebrated paper [3] (see also [4–6]), reading as

$$\begin{cases} u \in \mathrm{Fix}(G) \\ df|_{\mathrm{Fix}(G)}(u) = 0 \end{cases} \Longrightarrow df(u) = 0. \tag{1}$$

In other words, in order for an invariant point u to be critical for f, it is sufficient that it be critical for $f|_{\mathrm{Fix}(G)}$, namely if the directional derivative $f'(u)v$ vanishes along any direction v tangent to $\mathrm{Fix}(G)$, then it must vanish along the directions transverse to $\mathrm{Fix}(G)$ as well. Although the implication is valid in a rather broad context, there are of course some pathological counterexamples showing that it can fail to hold (see [3, Section 3]).

The validity of Palais' principle of symmetric criticality has a powerful impact on applications to nonlinear problems of mathematical physics which are set on unbounded domains (for instance $\mathbb{R}^N$, half-spaces or strips), are invariant under some linear transformations, such as rotations, thus exhibiting symmetry (for instance, spherical or cylindrical) and are associated with a suitable energy functional f. In fact, in this framework, it is often the case that the solutions of a PDE live

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in a Banach space X which is continuously embedded into a space Banach W but, unfortunately, the injection $X \hookrightarrow W$ fails to be compact. Typically the unboundedness of the domain where the problem is set prevents these inclusions from being compact and the problem cannot be solved by standard methods of nonlinear analysis. On the contrary, in many interesting concrete situations, for suitable compact Lie groups G, the compactness of the embedding $\mathrm{Fix}(G) \hookrightarrow W$ is restored. We refer the reader to the classical works by Lions [7] and Strauss [8] (see for instance [9, Section 1.5] for a list of the results of [7,8]). This fact offers a fruitful tool in order to find a critical point u of $f|_{\mathrm{Fix}(G)}$ at the Mountain Pass level (hence a critical point of f by the principle and, in turn, a solution to the associated problem, in a suitable sense). In the language of non-smooth critical point theory the implication (1) corresponds to the property

$$\begin{cases} u \in \mathrm{Fix}(G) \\ 0 \in \partial f|_{\mathrm{Fix}(G)}(u) \end{cases} \implies \quad 0 \in \partial f(u), \tag{2}$$

where $\partial f(u)$ is the subdifferential of f at u. In the current literature, some papers have already investigated the validity of this implication, for locally Lipschitz functions by Krawcewicz and Marzantowicz [10], for sums of C^1 (resp. locally Lipschitz) functions with lower semi-continuous convex functionals in the nice work by Kobayashi and Otani [11] (resp. by Kristaly et al. [12]). On the other hand, a suitable definition of $\partial f(u)$ for any function $f : X \to \bar{\mathbb{R}}$ has been recently given by Campa and Degiovanni in [13], consistently with the subdifferential of convex functions and containing the subdifferential in the sense of Clarke [14] for locally Lipschitz functions.

The main goal of this paper is to obtain two new Palais' symmetric criticality type principles for non-smooth functionals and furnish an application to elliptic PDEs associated with merely lower semi-continuous functionals, thus out of the frame of the previous literature available on the subject. In Section 2 we will recall some definitions and tools of non-smooth analysis. In Section 3 we provide a first non-smooth version of the symmetric criticality principle, Theorem 4, for a class of functionals which includes, in particular, the cases covered by the previous literature (cf. Proposition 3). Theorem 4 asserts that, under suitable assumptions, property (2) holds true. In Section 4 we shall obtain a second abstract result in a very general framework (Theorem 8). The result proves, under suitable assumptions, the implication

$$\begin{cases} u \in \mathrm{Fix}(G) \\ 0 \in \partial f|_{\mathrm{Fix}(G)}(u) \\ \exists\, V_u \subset X \text{ invariant, dense subspace} \end{cases} \implies \forall v \in V_u : f'(u)v = 0. \tag{3}$$

In fact, in many concrete situations, the classical directional derivative exists on suitable dense subspaces of X, and the criticality of u over these spaces is often sufficient to guarantee that, actually, u is the solution of a corresponding PDE, in a generalized or distributional sense (cf. [15]). In Section 6, we shall discuss an application (Theorem 12) of Theorem 8 showing that, for an unbounded domain $\Omega \subset \mathbb{R}^N$ invariant under a group G_N of rotations, under suitable assumptions there exists a nontrivial G_N-invariant solution to the quasi-linear problem

$$\begin{cases} \displaystyle\int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v + \int_\Omega |u|^{p-2}uv = \int_\Omega |u|^{q-2}uv, \\ \forall v \in C_c^\infty(\Omega). \end{cases}$$

As a consequence (Corollary 13), in the case $\Omega = \mathbb{R}^N$, we will prove that this problem admits a radial solution, extending [9, Theorems 1.29] to the quasi-linear case. We would like to stress that the previous non-smooth versions of Palais' symmetric criticality principle in the literature [11,12] cannot be used in order to prove the existence of solutions to the above elliptic problem, since its associated invariant functional is no more than lower semi-continuous (and nonconvex). For an overview on these classes of quasi-linear problems, we refer the interested reader to the monograph [16].

In contrast with the use of Palais' symmetric criticality principle discussed above, a direct approach is also possible, where, in the search of symmetric solutions, one avoids restricting the functional to $\mathrm{Fix}(G)$. This approach has been recently developed by the author [17] for a class of lower semi-continuous functionals, extending previous results from [18] valid for C^1 functionals. In Section 5 we state and prove a new abstract result, Theorem 9, yielding the existence of a Palais–Smale sequence $(u_h) \subset X$ for f at level c which is compact in a suitable Banach space W containing X, and it becomes more and more symmetric, as h increases. A final remark is now adequate. Let $\mathbb{D}$ and $\mathbb{S}$ be the closed unit ball and the sphere in $\mathbb{R}^m$ ($m \geq 1$), respectively, and $\Gamma_0 \subset C(\mathbb{S}, X)$, consider the (unrestricted) Mountain Pass energy level

$$c := \inf_{\gamma \in \Gamma} \sup_{\tau \in \mathbb{D}} f(\gamma(\tau)), \quad \Gamma := \big\{\gamma \in C(\mathbb{D}, X) : \gamma|_{\mathbb{S}} \in \Gamma_0\big\}.$$

Then, setting

$$\Gamma_0^{\mathrm{Fix}(G)} := \{\gamma \in \Gamma_0 : \gamma(\mathbb{S}) \subset \mathrm{Fix}(G)\},$$

one can consider, for $f|_{\mathrm{Fix}(G)} : \mathrm{Fix}(G) \to \bar{\mathbb{R}}$, the restricted Mountain Pass energy level

$$c^{\mathrm{Fix}(G)} := \inf_{\gamma \in \Gamma^{\mathrm{Fix}(G)}} \sup_{\tau \in \mathbb{D}} f(\gamma(\tau)), \quad \Gamma^{\mathrm{Fix}(G)} := \{\gamma \in C(\mathbb{D}, \mathrm{Fix}(G)) : \gamma|_{\mathbb{S}} \in \Gamma_0^{\mathrm{Fix}(G)}\}. \tag{4}$$

Then, automatically, it holds $c \leq c^{\mathrm{Fix}(G)}$ in light of $\Gamma^{\mathrm{Fix}(G)} \subset \Gamma$. Hence, in general, the strategy based upon Palais' criticality principle ensures a simpler approach to concrete problems but, as a drawback, the minimality property of the Mountain Pass energy level c might be lost, which is not the case in the above described direct approach.

2. Some notions from non-smooth analysis

In this section we consider abstract notions and results that will be used in the proof of the main result. For the definitions, we refer e.g. to [13,19,20]. Let X be a metric space, $B(u,\delta)$ the open ball of center u and of radius δ and let $f : X \to \bar{\mathbb{R}}$ be a function. We set

$$\mathrm{dom}(f) = \{u \in X : f(u) < +\infty\} \quad \text{and} \quad \mathrm{epi}\,(f) = \{(u,\xi) \in X \times \mathbb{R} : f(u) \le \xi\}.$$

We recall the definition of the weak slope.

Definition 1. For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of σ's in $[0,\infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : B((u, f(u)), \delta) \cap \mathrm{epi}(f) \times [0,\delta] \to X,$$

satisfying

$$d(\mathcal{H}((\xi,\mu),t),\xi) \le t, \qquad f(\mathcal{H}((\xi,\mu),t)) \le f(\xi) - \sigma t$$

for all $(\xi,\mu) \in B((u, f(u)),\delta) \cap \mathrm{epi}(f)$ and $t \in [0,\delta]$. The extended real number $|df|(u)$ is called the weak slope of f at u.

Remark 1. Let X be a metric space, $f : X \to \mathbb{R}$ a continuous function, and $u \in X$. Then $|df|(u)$ is the supremum of the real numbers σ in $[0,\infty)$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B(u,\delta) \times [0,\delta] \to X$, such that, for every v in $B(u,\delta)$, and for every t in $[0,\delta]$ it results

$$d(\mathcal{H}(v,t),v) \le t, \qquad f(\mathcal{H}(v,t)) \le f(v) - \sigma t.$$

If furthermore X is an open subset of a normed space E and f is a function of class C^1 on X, then $|df|(u) = \|df(u)\|$, for every $u \in X$ (see [20, Corollary 2.12]).

Let us define the function $\mathcal{G}_f : \mathrm{epi}\,(f) \to \mathbb{R}$ by

$$\mathcal{G}_f(u,\xi) = \xi. \tag{5}$$

In the following, epi (f) will be endowed with the metric

$$d((u,\xi),(v,\mu)) = (d(u,v)^2 + (\xi-\mu)^2)^{1/2},$$

so that the function $\mathcal{G}_f$ is Lipschitz continuous of constant 1.

We have the following

Proposition 1. *For every $u \in X$ such that $f(u) \in \mathbb{R}$, we have*

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u, f(u))^2}}, & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1,\\ +\infty, & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1.\end{cases}$$

In order to apply the abstract theory to the study of lower semi-continuous functions, the following condition is crucial

$$\forall (u,\xi) \in \mathrm{epi}\,(f) : f(u) < \xi \Longrightarrow |d\mathcal{G}_f|(u,\xi) = 1. \tag{6}$$

We refer the reader to [19,20] where this is discussed.

Now let X and X^* denote a real normed space and its topological dual, respectively. We recall [13, Definitions 4.3 and 5.5], namely, the definition of the generalized directional derivative.

Definition 2. Let $u \in X$ with $f(u) \in \mathbb{R}$. For every $v \in X$ and $\varepsilon > 0$ we define $f^\circ_\varepsilon(u; v)$ to be the infimum of the r's in $\mathbb{R}$ such that there exist $\delta > 0$ and a continuous function

$$\mathcal{V} : B_\delta(u, f(u)) \cap \mathrm{epi}(f) \times]0,\delta] \to B_\varepsilon(v),$$

which satisfies

$$f(\xi + t\mathcal{V}((\xi,\mu),t)) \le \mu + rt,$$

whenever $(\xi,\mu) \in B_\delta(u, f(u)) \cap \mathrm{epi}(f)$ and $t \in]0,\delta]$. Finally, we set

$$f^\circ(u; v) := \sup_{\varepsilon>0} f^\circ_\varepsilon(u; v) = \lim_{\varepsilon\to 0^+} f^\circ_\varepsilon(u; v).$$

We say that $f^\circ(u; v)$ is the directional derivative of f at u with respect to v.

It is readily seen that the directional derivative $f^\circ(u;v)$ does not change if the norm of X is substituted by an equivalent one.

According to [13, Definition 4.1, Corollary 4.7], we recall the notion of subdifferential.

Definition 3. Let $u \in X$ with $f(u) < +\infty$. We set

$$\partial f(u) := \left\{\alpha \in X^* : \langle \alpha, v\rangle \le f^\circ(u;v), \forall v \in X\right\}.$$

The set $\partial f(u)$ is convex and weakly* closed in X^*.

Again, the set $\partial f(u)$ does not change if the norm of X is substituted by an equivalent one.

We conclude the section by recalling [13, Corollary 4.13(ii)–(iii)], establishing the connection between the weak slope $|df|(u)$ and the subdifferential $\partial f(u)$.

Proposition 2. *Let $f : X \to \bar{\mathbb{R}}$ be a functional and $u \in X$ with $f(u) \in \mathbb{R}$. Assume that $|df|(u) < +\infty$. Then $\partial f(u) \neq \emptyset$ and*

$$\min\{\|\alpha\| : \alpha \in \partial f(u)\} \le |df|(u).$$

In particular, $|df|(u) = 0$ implies $0 \in \partial f(u)$.

The previous notions allow us to give the following.

Definition 4. We say that $u \in \mathrm{dom}(f)$ is a critical point of f if $0 \in \partial f(u)$. Moreover, $c \in \mathbb{R}$ is a critical value of f if there exists a critical point $u \in \mathrm{dom}(f)$ of f with $f(u) = c$.

Definition 5. Let $c \in \mathbb{R}$. We say that f satisfies the Palais–Smale condition at level c ($(PS)_c$ in short), if every sequence (u_n) in $\mathrm{dom}(f)$ such that $|df|(u_n) \to 0$ and $f(u_n) \to c$ as $n \to \infty$ admits a subsequence (u_{n_k}) converging in X.

3. A first abstract result

Let G be a compact Lie group acting linearly on a real Banach space $(X, \|\cdot\|)$. By suitably renorming X with an equivalent norm, we may assume without loss of generality that the action of G over X is isometric, that is

$$\forall u \in X,\ \forall g \in G: \quad \|gu\| = \|u\|.$$

For the proof, we refer the reader e.g. to [11, Proposition 3.15].

Now let μ denote the normalized *Haar measure* and define the map $A : X \to X$, known as *averaging map* or barycenter map, providing the center of gravity of the orbit of a $v \in X$, by

$$\forall v \in X: \quad Av := \int_G gv\,\mathrm{d}\mu(g).$$

The map A enjoys some useful properties. Firstly, $Av = v$ for all $v \in \mathrm{Fix}(G)$. Moreover, it is a continuous linear projection from X onto $\mathrm{Fix}(G)$, by the left invariance of μ. Finally, if $C \subseteq X$ is a closed, invariant (namely $gC \subseteq C$ for all $g \in G$), convex subset of X, then $A(C) \subseteq C$. See for instance [3, Section 5].

3.0.1. *The statement*

We consider the following assumption on f:

We assume that for all $u, v \in \mathrm{Fix}(G)$ there exist $\rho > 0$ and $C > 0$ with

$$\begin{cases} \dfrac{f(\zeta + t(z - Az)) - f(\zeta)}{t} \ge -C\|z - Az\|, \\ \forall \zeta \in B_\rho(u) \cap \mathrm{Fix}(G), \qquad \forall z \in B_\rho(v), \qquad \forall t \in]0, \rho] \quad \text{with } f(\zeta - tAz) \le f(u) + \rho. \end{cases} \tag{7}$$

As stated next, condition (7) includes, in particular, the classes of functions already considered in the previous literature [11,12] on the subject. More precisely, we have the following.

Proposition 3. *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be such that $f = f_0 + f_1$, where $f_0 : X \to \mathbb{R}$ is a locally Lipschitz function and $f_1 : X \to \mathbb{R} \cup \{+\infty\}$ is a convex, proper, lower semi-continuous and G-invariant function. Then condition (7) is satisfied.*

Proof. Condition (7) can be checked independently for f_0 and f_1. The proof for f_0 is trivial. Let us turn to the proof for f_1. For every $\zeta \in \mathrm{Fix}(G)$, $z \in X$ and $t \ge 0$, we consider the set

$$C := \left\{\eta \in X : f_1(\eta) \le f_1(\zeta + t(z - Az))\right\}.$$

Of course C is convex and closed, since f_1 is convex and lower semi-continuous. Moreover, C is G-invariant too, since f_1 is G-invariant. Then, $A(C) \subseteq C$. Recalling that $\zeta \in \mathrm{Fix}(G)$, we have $\zeta = A(\zeta + t(z - Az)) \in C$, namely $f_1(\zeta + t(z - Az)) - f_1(\zeta) \ge 0$, which yields (7). $\square$

The main result of the section is the following.

Theorem 4. *Let G be a compact Lie group acting linearly on X, let $f : X \to \bar{\mathbb{R}}$ be a G-invariant functional satisfying condition (7) and $u \in \mathrm{Fix}(G)$ with $f(u) \in \mathbb{R}$ be a critical point of $f|_{\mathrm{Fix}(G)}$. Then u is a critical point of f.*

The theorem is obtained in a quite simple fashion via the abstract machinery developed in [13] and which does not actually involve any a priori regularity assumption on the map f. Assumption (7) only enters in the proof of Proposition 7.

3.0.2. Some preliminary results

Next we state some preparatory results for Theorem 4.

Proposition 5. *Let G be a compact Lie group acting linearly on X, let $f : X \to \bar{\mathbb{R}}$ be a G-invariant functional and $u \in \mathrm{Fix}(G)$ with $f(u) \in \mathbb{R}$. Then*

$$\forall v \in X,\ \forall g \in G : \quad f^\circ(u; v) = f^\circ(u; gv).$$

Proof. As recalled, the action of G over X can be assumed to be isometric. Given $v \in X$ and $g \in G$, fix $\varepsilon > 0$. Let $r \in \mathbb{R}$ such that there exist $\delta > 0$ and a continuous function

$$\mathcal{V} : B_\delta(u, f(u)) \cap \mathrm{epi}(f) \times]0, \delta] \to B_\varepsilon(v),$$

according to Definition 2. Since $u \in \mathrm{Fix}(G)$ and f is G-invariant, if $(\xi, \mu) \in B_\delta(u, f(u)) \cap \mathrm{epi}(f)$, then also $(g\xi, \mu) \in B_\delta(u, f(u)) \cap \mathrm{epi}(f)$ as $\|g\xi - u\| = \|g\xi - gu\| = \|\xi - u\| \le \delta$ and $f(g\xi) = f(\xi) \le \mu$. Then, we are allowed to consider the continuous function

$$\hat{\mathcal{V}} : B_\delta(u, f(u)) \cap \mathrm{epi}(f) \times]0, \delta] \to B_\varepsilon(g^{-1}v),$$

defined by setting

$$\hat{\mathcal{V}}((\xi, \mu), t) := g^{-1}\mathcal{V}((g\xi, \mu), t)$$

for all $(\xi, \mu) \in B_\delta(u, f(u)) \cap \mathrm{epi}(f)$ and $t \in]0, \delta]$. It follows that $\hat{\mathcal{V}}$ is well defined as

$$\|\hat{\mathcal{V}}((\xi, \mu), t) - g^{-1}v\| = \|g^{-1}\mathcal{V}((g\xi, \mu), t) - g^{-1}v\| = \|\mathcal{V}((g\xi, \mu), t) - v\| \le \varepsilon.$$

Furthermore, it follows that

$$\begin{aligned} f(\xi + t\hat{\mathcal{V}}((\xi, \mu), t)) &= f(\xi + tg^{-1}\mathcal{V}((g\xi, \mu), t)) \\ &= f(g\xi + t\mathcal{V}((g\xi, \mu), t)) \le \mu + rt, \end{aligned}$$

for all $(\xi, \mu) \in B_\delta(u, f(u)) \cap \mathrm{epi}(f)$ and $t \in]0, \delta]$. By definition, it follows that $f_\varepsilon^\circ(u; g^{-1}v) \le r$. By the arbitrariness of r, $f_\varepsilon^\circ(u; g^{-1}v) \le f_\varepsilon^\circ(u; v)$. In a similar fashion, the opposite inequality follows as well. In conclusion, $f_\varepsilon^\circ(u; g^{-1}v) = f_\varepsilon^\circ(u; v)$ for every $\varepsilon > 0$. The assertion follows by letting $\varepsilon \to 0$. □

Now we have the following.

Proposition 6. *Let G be a compact Lie group acting linearly on X, let $f : X \to \bar{\mathbb{R}}$ be a G-invariant functional and $u \in \mathrm{Fix}(G)$ with $f(u) \in \mathbb{R}$. Assume that*

$$\forall v \in \mathrm{Fix}(G) : \quad f^\circ(u; v) \ge 0.$$

Then

$$\forall v \in X : \quad f^\circ(u; v) \ge 0.$$

Proof. Let $\varepsilon > 0$ and consider the set

$$C_\varepsilon = \{v \in X : f^\circ(u, v) \le -\varepsilon\}.$$

Assume, by contradiction, that $C_\varepsilon \neq \emptyset$. By [13, Corollary 4.6] the map $\{v \mapsto f^\circ(u; v)\}$ is convex and lower semi-continuous. Then C_ε is convex and closed. Furthermore C_ε is G-invariant in X by means of Proposition 5. In turn, $A(C_\varepsilon) \subset C_\varepsilon \cap \mathrm{Fix}(G)$. Since $A(C_\varepsilon) \neq \emptyset$, we deduce that $C_\varepsilon \cap \mathrm{Fix}(G) \neq \emptyset$, which yields a contradiction. By the arbitrariness of ε,

$$\{v \in X : f^\circ(u, v) < 0\} = \bigcup_{\varepsilon > 0} C_\varepsilon = \emptyset,$$

which proves the assertion. □

The next assertion is the crucial step in order to link the generalized directional derivatives of $f|_{\mathrm{Fix}(G)}$ with the generalized directional derivatives of f at a point $u \in \mathrm{Fix}(G)$.

Proposition 7. *Let G be a compact Lie group acting linearly on X, let $f : X \to \bar{\mathbb{R}}$ be a functional which satisfies (7) and $u \in \mathrm{Fix}(G)$ with $f(u) \in \mathbb{R}$. Then*

$$\forall v \in \mathrm{Fix}(G): \quad f^{\circ}(u; v) \geq (f|_{\mathrm{Fix}(G)})^{\circ}(u; v).$$

In particular, if

$$\forall v \in \mathrm{Fix}(G): \quad (f|_{\mathrm{Fix}(G)})^{\circ}(u; v) \geq 0,$$

then

$$\forall v \in \mathrm{Fix}(G): \quad f^{\circ}(u; v) \geq 0.$$

Proof. Let $v \in \mathrm{Fix}(G)$ and let $\varepsilon \in (0, \rho)$, where $\rho > 0$ is the number appearing in assumption (7) on f. Moreover, let $r \in \mathbb{R}$ be such that there exist $\delta > 0$ and a continuous function

$$\mathcal{V} : B_\delta(u, f(u)) \cap \mathrm{epi}(f) \times]0, \delta] \to B_\varepsilon(v),$$

according to Definition 2. Then, we choose a number

$$0 < \delta' < \min\left\{\delta, \frac{\rho}{1 + \varepsilon + \|v\|}\right\}, \tag{8}$$

and we define a map

$$\hat{\mathcal{V}} : B_{\delta'}(u, f(u)) \cap \mathrm{epi}(f) \cap (\mathrm{Fix}(G) \times \mathbb{R}) \times]0, \delta'] \to B_\varepsilon(v) \cap \mathrm{Fix}(G),$$

by setting

$$\hat{\mathcal{V}}((\xi, \mu), t) := A\mathcal{V}((\xi, \mu), t),$$

for all $(\xi, \mu) \in B_{\delta'}(u, f(u)) \cap \mathrm{epi}(f) \cap (\mathrm{Fix}(G) \times \mathbb{R})$ and $t \in]0, \delta']$. Notice that, as $\|A\| \leq 1$,

$$\|\hat{\mathcal{V}}((\xi, \mu), t) - v\| = \|A\mathcal{V}((\xi, \mu), t) - Av\| \leq \|A\| \|\mathcal{V}((\xi, \mu), t) - v\| \leq \varepsilon,$$

for all $(\xi, \mu) \in B_{\delta'}(u, f(u)) \cap \mathrm{epi}(f) \cap (\mathrm{Fix}(G) \times \mathbb{R})$ and $t \in]0, \delta']$, so that $\hat{\mathcal{V}}$ is well defined. We shall use assumption (7), applied with

$$\zeta := \xi + tAz, \quad z := \mathcal{V}((\xi, \mu), t), \; (\xi, \mu) \in B_{\delta'}(u, f(u)) \cap \mathrm{epi}(f) \cap \mathrm{Fix}(G) \times \mathbb{R}, \; t \in]0, \delta'].$$

Notice that $\zeta \in \mathrm{Fix}(G), f(\zeta - tAz) = f(\xi) \leq \mu \leq f(u) + \delta' < f(u) + \rho$ and

$$\begin{aligned} \|z - Az\| &= \|\mathcal{V}((\xi, \mu), t) - A\mathcal{V}((\xi, \mu), t)\| \\ &\leq \|\mathcal{V}((\xi, \mu), t) - v\| + \|Av - A\mathcal{V}((\xi, \mu), t)\| \\ &\leq \varepsilon(1 + \|A\|) \leq 2\varepsilon. \end{aligned} \tag{9}$$

Moreover, $\|z - v\| = \|\mathcal{V}((\xi, \mu), t) - v\| \leq \varepsilon < \rho$ and, by the choice in (8),

$$\|\zeta - u\| \leq \|\xi - u\| + t\|A\| \|z\| \leq \delta' + \delta' \|z\| \leq \delta'(1 + \varepsilon + \|v\|) < \rho.$$

Then, for $(\xi, \mu) \in B_{\delta'}(u, f(u)) \cap \mathrm{epi}(f) \cap (\mathrm{Fix}(G) \times \mathbb{R})$ and $t \in]0, \delta']$, by (7) and (9),

$$\begin{aligned} f(\xi + t\hat{\mathcal{V}}((\xi, \mu), t)) &= f(\xi + tA\mathcal{V}((\xi, \mu), t)) = f(\zeta) \\ &\leq f(\zeta + t(z - Az)) + Ct\|z - Az\| \\ &\leq f(\xi + t\mathcal{V}((\xi, \mu), t)) + 2C\varepsilon t \\ &\leq \mu + (r + 2C\varepsilon)t. \end{aligned}$$

It follows that $r + 2C\varepsilon \geq (f|_{\mathrm{Fix}(G)})^{\circ}_{\varepsilon}(u; v)$. By the arbitrariness of $r \in \mathbb{R}$, it follows that

$$f^{\circ}_{\varepsilon}(u; v) \geq (f|_{\mathrm{Fix}(G)})^{\circ}_{\varepsilon}(u; v) - 2C\varepsilon.$$

Finally, by the arbitrariness of ε, letting $\varepsilon \to 0^+$, we conclude that

$$f^{\circ}(u; v) \geq (f|_{\mathrm{Fix}(G)})^{\circ}(u; v),$$

which yields the assertion. □

3.0.3. Proof of Theorem 4

Let $u \in \mathrm{Fix}(G)$ be a critical point of the function $f|_{\mathrm{Fix}(G)}$, that is, $0 \in \partial(f|_{\mathrm{Fix}(G)})(u)$. Then $(f|_{\mathrm{Fix}(G)})^\circ(u; v) \geq 0$ for every $v \in \mathrm{Fix}(G)$, by Definition 3. In light of Proposition 7, we have $f^\circ(u; v) \geq 0$ for all $v \in \mathrm{Fix}(G)$. Finally, by Proposition 6, we get $f^\circ(u; v) \geq 0$ for all $v \in X$, namely $0 \in \partial f(u)$, so that u is a critical point of f. □

Remark 2. In the proof of Proposition 7, given the deformation $\mathcal{V}$, in order to build a new deformation $\hat{\mathcal{V}}$ that satisfies the desired properties, the idea is to compose $\mathcal{V}$ with A to ensure that the values of $\hat{\mathcal{V}}$ are projected into $\mathrm{Fix}(G)$. This, at the end, requires the technical assumption (7). In some concrete situations, the starting deformation $\mathcal{V}$, when restricted to $\mathrm{Fix}(G) \times \mathbb{R}$, automatically brings the values into $\mathrm{Fix}(G)$, so that one can take directly $\hat{\mathcal{V}} = \mathcal{V}|_{\mathrm{Fix}(G)\times\mathbb{R}}$. See, for instance, the argument in the proof of Lemma 16.

4. A second abstract result

Let G be a compact Lie group acting linearly on a real Banach space $(X, \|\cdot\|)$. We assume that for every $u \in \mathrm{Fix}(G)$ there exists a G-invariant dense vectorial subspace V_u of X such that the following conditions are satisfied

$$\forall u \in \mathrm{Fix}(G),\ \forall v \in V_u: \quad \text{the directional derivative } f'(u)v \text{ exists}, \tag{10}$$

$$\forall u \in \mathrm{Fix}(G),\ \forall v \in V_u \cap \mathrm{Fix}(G): \quad (f|_{\mathrm{Fix}(G)})^\circ(u; v) \leq f'(u)v, \tag{11}$$

$$\forall u \in \mathrm{Fix}(G),\ \forall v \in V_u: \quad f^\circ(u; v) \leq f'(u)v. \tag{12}$$

Furthermore, for every $u \in \mathrm{Fix}(G)$, we assume that

$$V_u = \bigcup_{j\in J} C_j, \quad C_j \text{ is convex, closed, } G\text{-invariant with } f'(u)|_{C_j} \text{ continuous}. \tag{13}$$

The main result of the section is the following.

Theorem 8. *Let G be a compact Lie group acting linearly on a Banach space X and let $f : X \to \bar{\mathbb{R}}$ be a G-invariant functional. Assume that conditions (10)–(13) are satisfied and let $u \in \mathrm{Fix}(G)$ with $f(u) \in \mathbb{R}$ be a critical point of $f|_{\mathrm{Fix}(G)}$. Then*

$$\forall v \in V_u: \quad f'(u)v = 0.$$

Furthermore, $\partial f(u) = \{0\}$ provided that $\partial f(u)$ is nonempty.

In the statement of Theorem 8 we are not explicitly assuming any global regularity on the functional f. In the last section of the paper we shall apply it to a class of (nonconvex) lower semi-continuous functionals. This achievement could not be reached through previous non-smooth versions of Palais' symmetric criticality principle in the literature as they require f to be the sum of a locally Lipschitz function with a lower semi-continuous convex function [12].

4.0.4. Proof of Theorem 8

Let $u \in \mathrm{Fix}(G)$ be a critical point of $f|_{\mathrm{Fix}(G)}$, $0 \in \partial(f|_{\mathrm{Fix}(G)})(u)$. Then $(f|_{\mathrm{Fix}(G)})^\circ(u; v) \geq 0$, for every $v \in \mathrm{Fix}(G)$. In particular, $(f|_{\mathrm{Fix}(G)})^\circ(u; v) \geq 0$, for every $v \in V_u \cap \mathrm{Fix}(G)$. In turn, in light of (11), by exploiting the linearity of the map $\{v \mapsto f'(u)v\}$, we get

$$\forall v \in V_u \cap \mathrm{Fix}(G): \quad f'(u)v = 0.$$

Now let $\varepsilon > 0$, $j \geq 1$ and consider the set

$$D^j_\varepsilon = \{v \in C_j : f'(u)v \leq -\varepsilon\}.$$

Assume, by contradiction, that $D^j_\varepsilon \neq \emptyset$. Of course, D^j_ε is convex, closed and G-invariant in X (recall that assumption (13) holds). In turn, $A(D^j_\varepsilon) \subset D^j_\varepsilon \cap \mathrm{Fix}(G)$. We deduce that $D^j_\varepsilon \cap \mathrm{Fix}(G) \neq \emptyset$, which yields a contradiction. By the arbitrariness of ε and j,

$$\{v \in V_u : f'(u)v < 0\} = \bigcup_{j\in J}\bigcup_{\varepsilon>0} D^j_\varepsilon = \emptyset,$$

yielding

$$\forall v \in V_u: \quad f'(u)v = 0. \tag{14}$$

This proves the first assertion. Concerning the second assertion, assume that $\partial f(u)$ is not empty and let $\alpha \in X^*$ with $\alpha \in \partial f(u)$. Then, in light of (12) and (14), we have

$$\forall v \in V_u: \quad \langle \alpha, v\rangle \leq f^\circ(u; v) \leq f'(u)v = 0.$$

Then,

$$\forall v \in V_u: \quad \langle \alpha, v\rangle = 0.$$

Taking into account that V_u is dense in X, we conclude that $\alpha = 0$. Hence $\partial f(u) \subseteq \{0\}$. □

5. A direct approach

The main goal of this section is that of showing how a direct (or, say, unrestricted) approach in the search of symmetric critical points can also be outlined. In this approach, the compactness of the Palais–Smale sequences in a suitable space (relevant for applications to PDEs) is achieved by exploiting the symmetry properties of the functional instead of restricting the functional to a space of symmetric functions. After recalling an abstract symmetrization framework due to [18], we shall state and prove a general abstract result by using the main result from [17].

5.1. Abstract symmetrization

We recall a definition from [18].

Let X and V be two Banach spaces and $S \subset X$. We consider two maps $* : S \to S$, $u \mapsto u^*$ (*symmetrization map*) and $h : S \times \mathcal{H}_* \to S$, $(u, H) \mapsto u^H$ (*polarization map*), where $\mathcal{H}_*$ is a path-connected topological space. We assume that the following conditions hold:

1. X is continuously embedded in V;
2. h is a continuous mapping;
3. for each $u \in S$ and $H \in \mathcal{H}_*$ it holds that $(u^*)^H = (u^H)^* = u^*$ and $u^{HH} = u^H$;
4. there exists a sequence (H_m) in $\mathcal{H}_*$ such that, for $u \in S$, $u^{H_1 \cdots H_m}$ converges to u^* in V;
5. for every $u, v \in S$ and $H \in \mathcal{H}_*$ it holds that $\|u^H - v^H\|_V \le \|u - v\|_V$.

Furthermore $* : S \to S$ can be extended to the whole space X by setting $u^* := (\Theta(u))^*$ for all $u \in X$, where $\Theta : (X, \|\cdot\|_V) \to (S, \|\cdot\|_V)$ is a Lipschitz function such that $\Theta|_S = \mathrm{Id}|_S$.

5.2. Compactness of Palais–Smale sequences

The main result of this section is the following.

Theorem 9. *Let X be a reflexive Banach space, $S \subset X$ and $* : S \to S$ a symmetrization satisfying the requirements of the abstract symmetrization framework. Let V and W be Banach spaces containing X such that*

(a) *the injections $X \overset{i}{\hookrightarrow} V \overset{i'}{\hookrightarrow} W$ are continuous.*

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function satisfying (6). Let $\mathbb{D}$ and $\mathbb{S}$ denote the closed unit ball and the sphere in $\mathbb{R}^m$ ($m \ge 1$) respectively and $\Gamma_0 \subset C(\mathbb{S}, X)$. Let us define

$$\Gamma = \left\{\gamma \in C(\mathbb{D}, X) : \gamma|_{\mathbb{S}} \in \Gamma_0\right\}.$$

Assume that

$$+\infty > c = \inf_{\gamma \in \Gamma} \sup_{\tau \in \mathbb{D}} f(\gamma(\tau)) > \sup_{\gamma_0 \in \Gamma_0} \sup_{\tau \in \mathbb{S}} f(\gamma_0(\tau)) = a,$$

and that the following conditions hold

(b) *for all $H \in \mathcal{H}_*$, for all $u \in S$: $f(u^H) \le f(u)$;*
(c) *for all $\gamma \in \Gamma$ there exist $\hat{\gamma} \in \Gamma$ and $H_0 \in \mathcal{H}_*$ such that*

$$\hat{\gamma}(\mathbb{D}) \subset S, \qquad f \circ \hat{\gamma} \le f \circ \gamma, \qquad \hat{\gamma}\,|_{\mathbb{S}}^{H_0} \in \Gamma_0.$$

Assume furthermore that

(d) *for each Palais–Smale sequence $(u_h) \subset X$ for f, (u_h) is bounded in X;*
(e) *for each Palais–Smale sequence $(u_h) \subset X$ for f, (u_h^*) converges in W.*

Then there exist $u \in X$ and a Palais–Smale sequence $(u_h) \subset X$ for f at level c such that

1. *$u_h \to u$ weakly in X, as $h \to \infty$;*
2. *$u_h \to u$ strongly in W, as $h \to \infty$;*
3. *if (4)–(5) of the abstract framework hold with W in place of V, then $u = u^*$ in W.*

The theorem states the existence, under suitable assumptions, of a Palais–Smale sequence (u_h) in X which is convergent in a Banach W larger than X to a symmetric element. In particular, of course, it does not claim the compactness of (u_h) in the original space X. Notice also that, in the cases where the space X is a Sobolev space defined over a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and V, W are subcritical $L^p(\Omega)$ spaces then, by Rellich Theorem, the injection $X \hookrightarrow W$ is compact and the stated strong convergence in W automatically follows from the weak convergence in X. On the other hand, this is not the case for PDEs which present loss of compactness, such as those set on an unbounded domain. In this sense, Theorem 9 allows to avoid restricting the concrete functional to the space of radial functions and then use Palais' symmetric criticality principle studied in the previous sections. It also provides an alternative to concentration-compactness arguments [21,22] under symmetry assumptions. It is quite easy to realize that, whenever the conclusion of the theorem holds, namely $u_h \to u$ weakly in X and strongly in W, then it is often the case that, in turn, $u_h \to u$ strongly in X (using $f(u_h) \to c$ and $|df|(u_h) \to 0$ as $h \to \infty$). See, for instance, the compactness argument inside the proof of Theorem 12. For an application of Theorem 9 in the case $p = 2$ and $f \in C^1$ (the semi-linear case), see [18, Theorem 4.5]. In a similar fashion, applications to lower semi-continuous functionals can be given, by arguing as in [17], up to suitable necessary modifications.

5.3. A concrete framework

As a meaningful concrete framework where Theorem 9 applies one can think, for instance, of the case where, for $p < m < p^*$,

$$W^{1,p}(\mathbb{R}^N) = S = X \overset{i}{\hookrightarrow} V = L^p \cap L^{p^*}(\mathbb{R}^N) = V \overset{i'}{\hookrightarrow} W = L^m(\mathbb{R}^N),$$

with i, i' continuous injections. The polarization and symmetrization functions are defined as $u^H = |u|^H$ and $u^* = |u|^*$. Given x in $\mathbb{R}^N$ and a polarizer H (half-space) the reflection of x with respect to the boundary of H is denoted by x_H. The polarization of $u : \mathbb{R}^N \to \mathbb{R}^+$ by a polarizer H is the function $u^H : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\}, & \text{if } x \in H \\ \min\{u(x), u(x_H)\}, & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{15}$$

The Schwarz symmetrization of a set $C \subset \mathbb{R}^N$ is the unique open ball centered at the origin C^* such that $\mathcal{L}^N(C^*) = \mathcal{L}^N(C)$, $\mathcal{L}^N$ being the N-dimensional outer Lebesgue measure. A measurable function u is admissible for the Schwarz symmetrization if it is nonnegative and, for every $\varepsilon > 0$, the Lebesgue measure of $\{u > \varepsilon\}$ is finite. The Schwarz symmetrization of an admissible function $u : C \to \mathbb{R}^+$ is the unique function $u^* : C^* \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$, it holds that $\{u^* > t\} = \{u > t\}^*$. Then (1)–(5) in the abstract framework are satisfied (cf. [18]). One could also consider the case $S \subset X$, by setting $X = W^{1,p}(\mathbb{R}^N)$, $S = W^{1,p}_+(\mathbb{R}^N)$, $V = L^p \cap L^{p^*}(\mathbb{R}^N)$ and u^H, u^* defined as usual for $u \in S$ (and u^* extends to $X \setminus S$ by setting $u^* = |u|^*$). Also in this case, (1)–(5) are satisfied [18]. In both cases (4)–(5) also hold when substituting V with W.

5.4. Proof of Theorem 9

In [17], the author proved the following.

Theorem 10. *Let X and V be two Banach spaces, $S \subset X$, $*$ and $\mathcal{H}_*$ satisfying the requirements of the abstract symmetrization framework. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function satisfying (6). Let $\mathbb{D}$ and $\mathbb{S}$ denote the closed unit ball and the sphere in $\mathbb{R}^m$ ($m \geq 1$), respectively, and $\Gamma_0 \subset C(\mathbb{S}, X)$. Let us define*

$$\Gamma = \left\{\gamma \in C(\mathbb{D}, X) : \gamma|_{\mathbb{S}} \in \Gamma_0\right\}.$$

Assume that

$$+\infty > c = \inf_{\gamma\in\Gamma} \sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) > \sup_{\gamma_0\in\Gamma_0} \sup_{\tau\in\mathbb{S}} f(\gamma_0(\tau)) = a,$$

and that

$$\forall H \in \mathcal{H}_*,\ \forall u \in S: \quad f(u^H) \leq f(u).$$

Then, for every $\varepsilon \in (0, (c-a)/3)$, every $\delta > 0$ and $\gamma \in \Gamma$ such that

$$\sup_{\tau\in\mathbb{D}} f(\gamma(\tau)) \leq c + \varepsilon, \qquad \gamma(\mathbb{D}) \subset S, \qquad \gamma|_{\mathbb{S}}^{H_0} \in \Gamma_0 \quad \text{for some } H_0 \in \mathcal{H}_*,$$

there exists $u \in X$ such that

$$c - 2\varepsilon \leq f(u) \leq c + 2\varepsilon, \qquad |df|(u) \leq 3\varepsilon/\delta, \qquad \|u - u^*\|_V \leq 3((1 + C_\Theta)K + 1)\delta, \tag{16}$$

K being the norm of the embedding map $i : X \to V$ and C_Θ the Lipschitz constant of Θ.

5.5. Proof of Theorem 9 concluded

By the definition of the minimax value c, by choosing $\varepsilon = \varepsilon_h = h^{-2}$, for each $h \geq 1$ there exists a curve $\gamma_h \in \Gamma$ such that

$$\sup_{\tau\in\mathbb{D}} f(\gamma_h(\tau)) \leq c + \frac{1}{h^2}.$$

Then, by virtue of assumption (c), we can find a curve $\hat{\gamma}_h \in \Gamma$ and $H_0^h \in \mathcal{H}_*$ such that

$$\sup_{\tau\in\mathbb{D}} f(\hat{\gamma}_h(\tau)) \leq c + \frac{1}{h^2}, \qquad \hat{\gamma}_h(\mathbb{D}) \subset S, \qquad \hat{\gamma}_h|_{\mathbb{S}}^{H_0^h} \in \Gamma_0.$$

Choose also $\delta = \delta_h = h^{-1}$. In turn, recalling assumption (b), by Theorem 10 there exists a sequence $(u_h) \subset X$ such that $f(u_h) \to c$ and $|df|(u_h) \to 0$ as $h \to \infty$, with the additional information that

$$\|u_h - u_h^*\|_V \to 0, \quad \text{as } h \to \infty. \tag{17}$$

In particular, by Definition 5, (u_h) is a Palais–Smale sequence for f at the level c. By assumption (d), (u_h) is bounded in X. Since X is reflexive, there exists $u \in X$ such that, up to a subsequence, $u_h \to u$ weakly in X, as $h \to \infty$. Now consider the sequence (u_h^*) of abstract symmetrizations of u_h. By assumptions (a) and (e), it follows that (u_h^*) converges strongly in W and V is continuously embedded into W. In particular, there exists $v \in X$ such that $u_h^* \to v$ in W as $h \to \infty$ and, for all $h \geq 1$,

$$\|u_h - v\|_W \leq \|u_h - u_h^*\|_W + \|u_h^* - v\|_W \leq C\|u_h - u_h^*\|_V + \|u_h^* - v\|_W,$$

which yields $u_h \to v$ strongly in W, for $h \to \infty$, on account of (17). Then, of course, we deduce that $v = u$, which yields $u_h \to u$ strongly in W. Finally, assume that conditions (4)–(5) of the abstract symmetrization framework holds also with W in place of V. Then it is readily seen that $\|z^* - w^*\|_W \leq C_\Theta\|z - w\|_W$ for all $z, w \in X$ (cf. [17, Remark 2.1]). In turn, for all $h \geq 1$, we get

$$\begin{aligned}\|u^* - u\|_W &\leq \|u^* - u_h^*\|_W + \|u_h^* - u_h\|_W + \|u_h - u\|_W \\ &\leq (1 + C_\Theta)\|u_h - u\|_W + C\|u_h^* - u_h\|_V,\end{aligned}$$

which yields $u = u^*$ in W, by virtue of (17) and the limit $u_h \to u$ in W. □

6. Application to quasi-linear PDEs

In this section, we shall consider some applications of the abstract Palais criticality principle developed in Section 4.

6.0.1. *Compatible invariant domains*

In the following, the symbol G_N will denote a subgroup of $\mathcal{O}(N)$, the orthogonal group over $\mathbb{R}^N$, with $N \geq 2$. According to [9, Definition 1.22], we consider the following.

Definition 6. For every $y \in \mathbb{R}^N$ and $r > 0$, we set

$$\mathscr{M}(y, r, G_N) := \sup\left\{n \in \mathbb{N} : \exists g_1, \ldots, g_n \in G_N : i \neq j \Rightarrow B_r(g_i y) \cap B_r(g_j y) = \emptyset\right\}.$$

An open (possibly unbounded) subset Ω of $\mathbb{R}^N$ is said to be invariant provided that $g\Omega = \Omega$, for all $g \in G_N$. An invariant subset Ω is said to be compatible with G_N provided that

$$\lim_{\substack{|y|\to\infty \\ \operatorname{dist}(y,\Omega)\leq r}} \mathscr{M}(y, r, G_N) = \infty$$

for some positive number r.

Throughout the rest of the paper we shall assume that $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ is a (possibly unbounded) smooth domain which is invariant under the action of G_N. Let

$$X = W_0^{1,p}(\Omega), \quad 1 < p < N, \qquad \|u\|_{1,p} = \left\{\|u\|_p^p + \|Du\|_p^p\right\}^{1/p}.$$

The action of G_N on $W_0^{1,p}(\Omega)$ is defined in a standard fashion by setting

$$\forall u \in W_0^{1,p}(\Omega),\ \forall g \in G_N : \quad (gu)(x) := u(g^{-1}x) \quad \text{for a.e. } x \in \Omega.$$

We shall denote by $\mathrm{Fix}(G_N)$ the set of fixed points u of X with respect to the action of G_N, namely $gu = u$ for all $g \in G_N$.

By virtue of [11, Proposition 4.2] we have the following.

Lemma 11. *Assume that Ω is compatible with G_N. Then the embeddings*

$$\mathrm{Fix}(G_N) \hookrightarrow L^m(\Omega), \quad \text{for all } p < m < p^* \tag{18}$$

are compact.

Remark 3. A particular, but important, case is contained in Lemma 11, namely

$$\Omega = \mathbb{R}^N, \qquad G_N := \mathscr{O}(N_1) \times \mathscr{O}(N_2) \times \cdots \mathscr{O}(N_\ell), \quad N = \sum_{j=1}^{\ell} N_j, \ \ell \geq 1, \ N_j \geq 2.$$

In fact, $\mathbb{R}^N$ is compatible with this G_N (cf. [9, Corollary 1.25]). Then Lemma 11 yields

$$\mathrm{Fix}_{W^{1,p}(\mathbb{R}^N)}(\mathscr{O}(N_1) \times \mathscr{O}(N_2) \times \cdots \mathscr{O}(N_\ell)) \hookrightarrow L^m(\Omega), \quad \text{for all } p < m < p^*,$$

with compact injection. In particular, $\mathrm{Fix}_{W^{1,p}(\mathbb{R}^N)}(\mathscr{O}(N)) \hookrightarrow L^m(\Omega)$ with compact injection.

6.0.2. Invariant quasi-linear functionals

Let $f : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the functional

$$f(u) = J(u) + I(u), \quad J(u) := \int_\Omega j(u, |Du|), \ I(u) := \int_\Omega \frac{|u|^p}{p} - \int_\Omega \frac{|u|^q}{q}, \tag{19}$$

where $p < q < p^*$. Notice that $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a C^1 functional while, on the contrary, $J : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ is, in general, merely lower semi-continuous, by Fatou's Lemma, if $j \geq 0$. Moreover, f is invariant under G_N, being invariant under the action of $\mathscr{O}(N)$, that is

$$\forall u \in W_0^{1,p}(\Omega), \ \forall g \in \mathscr{O}(N) : \quad f(gu) = f(u).$$

We consider the following assumptions on j. We assume that, for every s in $\mathbb{R}$,

$$\big\{t \mapsto j(s,t)\big\} \text{ is strictly convex and increasing.} \tag{20}$$

Moreover, there exist a constant $\alpha_0 > 0$ and a positive increasing function $\alpha \in C(\mathbb{R})$ such that, for every $(s,t) \in \mathbb{R} \times \mathbb{R}^+$, it holds that

$$\alpha_0 t^p \leq j(s,t) \leq \alpha(|s|) t^p. \tag{21}$$

The functions $j_s(s,t)$ and $j_t(s,t)$ denote the derivatives of $j(s,t)$ with respect to the variables s and t, respectively. Regarding the function $j_s(s,t)$, we assume that there exist two positive increasing functions $\beta, \gamma \in C(\mathbb{R})$ and a positive constant R such that

$$|j_s(s,t)| \leq \beta(|s|) t^p, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and all } t \in \mathbb{R}^+, \tag{22}$$

$$|j_t(s,t)| \leq \gamma(|s|) t^{p-1}, \quad \text{for every } s \text{ in } \mathbb{R} \text{ and all } t \in \mathbb{R}^+, \tag{23}$$

$$j_s(s,t)s \geq 0, \quad \text{for every } s \text{ in } \mathbb{R} \text{ with } |s| \geq R \text{ and all } t \in \mathbb{R}^+. \tag{24}$$

It is readily seen that, without loss of generality, we can assume that $\gamma = \alpha$, up to a constant. Furthermore, we assume that there exist $R' > 0$ and $\delta > 0$ such that

$$qj(s,t) - j_s(s,t)s - (1+\delta) j_t(s,t)t \geq 0, \quad \text{for every } s \in \mathbb{R} \text{ with } |s| \geq R' \tag{25}$$

and all $t \in \mathbb{R}^+$. Finally, it holds that

$$\lim_{|s| \to \infty} \frac{\alpha(|s|)}{|s|^{q-p}} = 0. \tag{26}$$

In the above assumptions, f is merely lower semi-continuous. If α is bounded, then f becomes a continuous functional. Condition (24) is typical for these problems and plays a significant rǒle in the verification of the Palais–Smale condition and in the regularity theory (cf. e.g. [16]). Condition (25) allows the Palais–Smale sequences to be bounded in $W_0^{1,p}(\Omega)$, while (26) guarantees that f admits a Mountain Pass geometry.

6.0.3. Statement of the results

The main result of this section is the following.

Theorem 12. *Assume that Ω is compatible with G_N. Then there exists a nontrivial solution $u \in \mathrm{Fix}(G_N)$ to the quasi-linear problem*

$$\int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|) v + \int_\Omega |u|^{p-2} uv = \int_\Omega |u|^{q-2} uv, \tag{27}$$

for every $v \in C_c^\infty(\Omega)$.

Furthermore, let $\Omega = \mathbb{R}^N$ with $N \geq 2$. Then, we have the following.

Corollary 13. *Problem (27) admits a nontrivial radial positive solution.*

Remark 4. Assume that $p = 2$ and $j(s,t) = \frac{t^2}{2}$. Then, Corollary 13 reduces to the results due to Strauss [8] (see, for instance, [9, Theorem 1.29]).

Remark 5. In place of $|u|^{q-2}u$, more general nonlinearities $f(|x|, u)$ could be handled by Theorem 12. For instance, if $\Omega = \mathbb{R}^N$ and $p < q < p^*$, one could assume that, for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(|x|, s)| \leq \varepsilon |s|^{p-1} + C_\varepsilon |s|^{q-1}, \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and all } s \in \mathbb{R}. \tag{28}$$

Let $W^{1,p}_G(\mathbb{R}^N)$ denote $\mathrm{Fix}(G)$ when $X = W^{1,p}(\mathbb{R}^N)$. We claim that the map $\{u \mapsto f(|x|, u)\}$ is completely continuous from $W^{1,p}_G(\mathbb{R}^N)$ to its dual, as soon as the injection of $W^{1,p}_G(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$ is compact. In fact, let $(u_h) \subset W^{1,p}_G(\mathbb{R}^N)$ be a bounded sequence. Then, up to a subsequence, (u_h) converges weakly in $W^{1,p}(\mathbb{R}^N)$ and strongly in $L^q(\mathbb{R}^N)$ to some function $u \in W^{1,p}_G(\mathbb{R}^N)$. Let $\varepsilon > 0$ and let C_ε be the constant appearing in (28). Then, it is readily checked that, for all $\eta > 0$, there exist $R = R(\eta) > 0$ and $C > 0$ (independent of ε and η) with

$$\sup_{h \geq 1} \sup_{\substack{v \in W^{1,p}_G(\mathbb{R}^N) \\ \|v\|_{1,p} \leq 1}} \left| \int_{\mathbb{R}^N \setminus B(0,R)} (f(|x|, u_h) - f(|x|, u))v \right| < C_\varepsilon \eta + \varepsilon C.$$

Therefore, taking into account that $\|f(|x|, u_h) - f(|x|, u)\|_{L^{q'}(B(0,R))} \to 0$ as $h \to \infty$, the desired claim follows by letting first $h \to \infty$, then $\eta \to 0^+$ and, finally, $\varepsilon \to 0^+$.

6.0.4. Some preparatory facts

For all $u \in W^{1,p}_0(\Omega)$, define the space

$$V_u = \left\{ v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) : u \in L^\infty(\{x \in \Omega : v(x) \neq 0\}) \right\}. \tag{29}$$

The vector space V_u is dense in $W^{1,p}_0(\Omega)$ (cf. [23]). The following proposition, easy to prove, shows that V_u is a good test space to differentiate non-smooth functionals under suitable growth conditions. In particular, as a consequence, the abstract condition (10) is fulfilled.

Proposition 14. *Assume conditions (21), (22) and (24). Then, for every $u \in W^{1,p}_0(\Omega)$ with $J(u) < +\infty$ and every $v \in V_u$ we have*

$$j_s(u, |Du|)v \in L^1(\Omega), \qquad j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv \in L^1(\Omega),$$

with the agreement that $j_t(u, |Du|)\frac{Du}{|Du|} = 0$ when $|Du| = 0$ (in view of (24)). Moreover, the function $\{t \mapsto J(u + tv)\}$ is of class C^1 and

$$J'(u)(v) = \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v.$$

In particular,

$$f'(u)(v) = \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v + \int_\Omega |u|^{p-2}uv - \int_\Omega |u|^{q-2}uv,$$

for every $v \in V_u$.

For all $u \in \mathrm{Fix}(G_N)$ and any $j \geq 1$, let us now set

$$C_j = \{v \in W^{1,p}_0(\Omega) : |v| \leq j \text{ a.e. in } \Omega \text{ and } |u| \leq j \text{ a.e. where } v \neq 0\}.$$

Then, we have the following.

Lemma 15. *For all $u \in \mathrm{Fix}(G_N)$ and $j \geq 1$, the set C_j is convex, closed, G_N-invariant with $f'(u)|_{C_j}$ continuous and*

$$V_u = \bigcup_{j=1}^\infty C_j. \tag{30}$$

In particular V_u is G_N-invariant.

Proof. Of course each C_j is convex. Moreover C_j is G-invariant. In fact, if $v \in C_j$, then $|(gv)(x)| = |v(g^{-1}x)| \leq j$ a.e. in Ω, being the domain invariant under G_N. Moreover, if $x \in \{gv \neq 0\}$, then $v(g^{-1}x) \neq 0$, so that $|u(x)| = |u(g^{-1}x)| \leq j$, since $u \in \mathrm{Fix}(G_N)$. Hence $gv \in C_j$. Let us now prove that C_j is closed. If $(v_h) \subset C_j$ with $v_h \to v$ in $W_0^{1,p}(\Omega)$, then up to a subsequence $|v_h(x)| \leq j$ and $v_h(x) \to v(x)$, as $h \to \infty$, yielding $|v| \leq j$ a.e. in Ω. Moreover, if $A = \{v \neq 0\}$, then by the pointwise convergence,

$$A \subset B = \bigcup_{h=1}^{\infty} \{v_h \neq 0\}.$$

Since $\sup_B |u| \leq j$, it holds that $\sup_A |u| \leq \sup_B |u| \leq j$. Thus, each C_j is closed. Let us now prove (30). Let $v \in V_u$. Since $v \in L^\infty(\Omega)$, there exists $j_0 \geq 1$ such that $|v| \leq j_0$, a.e. in Ω. Since u is uniformly bounded over $\{v \neq 0\}$, up to enlarging j_0, we may as well assume that $|u| \leq j_0$ over $\{v \neq 0\}$. Hence $u \in C_{j_0}$, proving the assertion, the converse inclusion being trivial. Finally, let us prove that $f'(u)|_{C_j}$ is continuous. To this end, let $(v_h) \subset C_j$ with $v_h \to v$ in $W_0^{1,p}(\Omega)$, as $h \to \infty$. For all $h \geq 1$, since $v_h \in V_u$, by Proposition 14, we get

$$f'(u)v_h = \int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv_h + \int_\Omega j_s(u, |Du|) v_h + \int_\Omega |u|^{p-2} u v_h - \int_\Omega |u|^{q-2} u v_h.$$

Of course, the last two integrals converge to $\int_\Omega |u|^{p-2}uv$ and $\int_\Omega |u|^{q-2}uv$, respectively. Also

$$j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv_h \to j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv, \quad \text{a.e. in } \Omega,$$
$$j_s(u, |Du|) v_h \to j_s(u, |Du|) v, \quad \text{a.e. in } \Omega.$$

In addition, in light of the growth conditions on j_t, j_s, for some positive constant M_j,

$$\left| j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv_h \right| \leq M_j |Du|^{p-1} |Dv_h|, \quad \text{a.e. in } \Omega,$$
$$|j_s(u, |Du|) v_h| \leq M_j |Du|^p |v_h| \leq j M_j |Du|^p, \quad \text{a.e. in } \Omega.$$

Then, Lebesgue dominated convergence theorem yields $f'(u)v_h \to f'(u)v$, as $h \to \infty$. □

Then we have the following.

Lemma 16. *Assume conditions (21)–(23) and let $u \in \mathrm{Fix}(G_N)$ such that $J(u) < +\infty$. Then, the following facts hold:*

(i) *for every $v \in V_u$, we have*

$$J^\circ(u; v) \leq J'(u)v$$

(ii) *for every $v \in V_u \cap \mathrm{Fix}(G_N)$, we have*

$$(J|_{\mathrm{Fix}(G_N)})^\circ(u; v) \leq J'(u)v.$$

Proof. The proofs of assertions (i) and (ii) are similar. Hence, let us focus on the proof of (ii). Let $\eta > 0$ with $J(u) < \eta$. Moreover, let $v \in V_u \cap \mathrm{Fix}(G_N)$ and $\varepsilon > 0$. Now take $r \in \mathbb{R}$ with

$$J'(u)v = \int_\Omega j_t(u, |Du|) \frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|) v < r. \tag{31}$$

Let $H \in C^\infty(\mathbb{R})$ be a cut-off function such that $H(s) = 1$ on $[-1, 1]$, $H(s) = 0$ outside $[-2, 2]$ and $|H'(s)| \leq 2$ on $\mathbb{R}$. Notice that, as $v \in V_u$, we also have $H(\frac{u}{k})v \in V_u$ and $H(\frac{z}{k})v \in V_z$ for all $z \in W_0^{1,p}(\Omega)$ and $k \geq 1$. Then, by exploiting the growth conditions on j_t and j_s, it is possible to prove that there exist $k \geq 1$ and $\delta > 0$ (depending upon k) such that

$$\left\| H\left(\frac{z}{k}\right) v - v \right\|_{1,p} < \varepsilon, \tag{32}$$

as well as

$$J'\left(z + \vartheta H\left(\frac{z}{k}\right) v\right)\left(H\left(\frac{z}{k}\right) v\right) < r, \tag{33}$$

for all $z \in B(u, \delta) \cap J^\eta$ and $\vartheta \in [0, \delta)$. Since the map $\{t \mapsto J(z + tH(\frac{z}{k})v)\}$ is of class C^1, by applying Lagrange theorem on $[0, t]$ and taking into account (33), there exists $\vartheta \in [0, t]$ with

$$J\left(z + tH\left(\frac{z}{k}\right) v\right) - J(z) = tJ'\left(z + \vartheta H\left(\frac{z}{k}\right) v\right)\left(H\left(\frac{z}{k}\right) v\right) \leq rt, \tag{34}$$

for every $z \in B(u, \delta) \cap J^\eta$ and all $t \in [0, \delta)$. Up to reducing the value of $\delta > 0$, we may also assume that $J(u) + \delta < \eta$. Notice that $H(\frac{z}{k})v \in \mathrm{Fix}(G_N)$ for all $v, z \in \mathrm{Fix}(G_N)$, since

$$\forall g \in G_N : g\left(H\left(\frac{z}{k}\right)v\right) = H\left(\frac{gz}{k}\right)gv = H\left(\frac{z}{k}\right)v.$$

Then, on account of inequality (32), we are allowed to define the continuous function

$$\mathcal{H} : B_\delta(u, J(u)) \cap \mathrm{epi}(J) \cap (\mathrm{Fix}(G_N) \times \mathbb{R}) \times]0, \delta] \to B_\varepsilon(v) \cap \mathrm{Fix}(G_N),$$

by setting

$$\mathcal{H}((z, \mu), t) := H\left(\frac{z}{k}\right)v.$$

Notice that, for all $(z, \mu) \in B_\delta(u, J(u)) \cap \mathrm{epi}(J)$, we have $z \in B(u, \delta) \cap J^\eta$. Hence, by inequality (34), we have

$$J(z + t\mathcal{H}((z, \mu), t)) \le J(z) + rt \le \mu + rt$$

whenever $(z, \mu) \in B_\delta(u, J(u)) \cap \mathrm{epi}(J)$ and $t \in]0, \delta]$. Then, according to Definition 2, we can conclude that $(J|_{\mathrm{Fix}(G_N)})^\circ_\varepsilon(u; v) \le r$. By the arbitrariness of r, it follows that

$$(J|_{\mathrm{Fix}(G_N)})^\circ_\varepsilon(u; v) \le \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v.$$

By the arbitrariness of ε, we get $(J|_{\mathrm{Fix}(G_N)})^\circ(u; v) \le J'(u)v$, for all $v \in V_u$. □

Remark 6. Under the assumptions of Lemma 16, assuming that $\partial J(u) \neq \emptyset$, $\partial J(u) = \{\alpha\}$ for some $\alpha \in W^{-1,p'}(\Omega)$ such that

$$\forall v \in V_u : \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v = \langle \alpha, v \rangle.$$

On account of the definition of $\partial f(u)$, it is sufficient to combine (i) of Lemma 16, the linearity of $\{v \mapsto J'(u)v\}$ and, of course, the density of V_u in $W_0^{1,p}(\Omega)$. In a similar fashion, assuming $\partial J|_{\mathrm{Fix}(G_N)}(u) \neq \emptyset$, then $\partial J|_{\mathrm{Fix}(G_N)}(u) = \{\beta\}$ for some $\beta \in (\mathrm{Fix}(G_N))'$ such that

$$\forall v \in V_u \cap \mathrm{Fix}(G_N) : \quad \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v = \langle \beta, v \rangle.$$

This follows by (ii) of Lemma 16, the linearity of $\{v \mapsto J'(u)v\}$ and, finally, by the density of $V_u \cap \mathrm{Fix}(G_N)$ in $\mathrm{Fix}(G_N)$, which is stated in Proposition 20.

Finally, returning to the functional f defined in (19), we have the following.

Corollary 17. *Assume conditions (21)–(23) and let $u \in \mathrm{Fix}(G_N)$ such that $f(u) < +\infty$. Then, the following facts hold:*

(i) *for every $v \in V_u$, we have*

$$f^\circ(u; v) \le f'(u)v.$$

(ii) *for every $v \in V_u \cap \mathrm{Fix}(G_N)$, we have*

$$(f|_{\mathrm{Fix}(G_N)})^\circ(u; v) \le f'(u)v.$$

Proof. By (19), it is $f = J + I$ with $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ of class C^1. Then $I^\circ(u; v) = I'(u)v$ for any $v \in V_u$ and $(I|_{\mathrm{Fix}(G_N)})^\circ(u; v) = I'(u)v$ for any $v \in V_u \cap \mathrm{Fix}(G_N)$. Of course, we have $J(u) < +\infty$. Then, by combining [13, Theorem 5.1] with Lemma 16, we have

$$\begin{aligned} \forall v \in V_u : \quad f^\circ(u; v) &= (J + I)^\circ(u; v) \\ &\le J^\circ(u; v) + I^\circ(u; v) \\ &\le J'(u)v + I'(u)v = f'(u)v. \end{aligned}$$

In a similar fashion, we have

$$\begin{aligned} \forall v \in V_u \cap \mathrm{Fix}(G_N) : \quad (f|_{\mathrm{Fix}(G_N)})^\circ(u; v) &= (J|_{\mathrm{Fix}(G_N)} + I|_{\mathrm{Fix}(G_N)})^\circ(u; v) \\ &\le (J|_{\mathrm{Fix}(G_N)})^\circ(u; v) + (I|_{\mathrm{Fix}(G_N)})^\circ(u; v) \\ &\le J'(u)v + I'(u)v = f'(u)v, \end{aligned}$$

concluding the proof. □

In light of the previous facts, we have the following.

Corollary 18. *Let $f : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be the G_N-invariant functional defined in (19) which satisfies the assumptions indicated in Section 6.0.2. Let $u \in \mathrm{Fix}(G_N)$ with $f(u) \in \mathbb{R}$ be a critical point of $f|_{\mathrm{Fix}(G_N)}$, that is $0 \in \partial f|_{\mathrm{Fix}(G_N)}(u)$. Then*

$$\forall v \in V_u : \quad f'(u)v = 0.$$

Furthermore, $\partial f(u) = \{0\}$ provided that $\partial f(u)$ is nonempty.

Proof. It is sufficient to apply Theorem 8, since conditions (10)–(13) are satisfied in view of Proposition 14, Lemma 15 and Corollary 17. □

Next we state some technical facts, necessary for the proof of the main result, Theorem 12.

Proposition 19. *It holds that*

$$\forall (u, \xi) \in \mathrm{epi}\,(f|_{\mathrm{Fix}(G_N)}) : \quad f(u) < \xi \Longrightarrow |d\mathcal{G}_{f|_{\mathrm{Fix}(G_N)}}|(u, \xi) = 1.$$

Proof. Notice first that [15, Theorem 3.11] extends to the case where $H_0^1(\Omega)$ is substituted by $W_0^{1,p}(\Omega)$, Ω is possibly unbounded and the growths (21)–(23) are assumed. Since $f = J + I$ with $I \in C^1(\mathrm{Fix}(G_N), \mathbb{R})$, the assertion follows by [15, Proposition 3.7] with $X = \mathrm{Fix}(G_N)$ by using [15, Theorem 3.11] with J replaced by the restriction $J|_{\mathrm{Fix}(G_N)}$. In fact, it is sufficient to notice that for the deformation (for $\delta, \eta > 0$ and $k \geq 1$)

$$\mathscr{H}(z, t) = (1 - t)z + tT_k(z), \quad z \in B(u, \delta) \cap J^\eta, \ t \in [0, \delta],$$

making the job in [15, Theorem 3.11] it is $\mathscr{H}(z, t) \in \mathrm{Fix}(G_N)$ for $z \in B(u, \delta) \cap \mathrm{Fix}(G_N)$. □

Proposition 20. *Let $u \in \mathrm{Fix}(G_N)$. Then the subspace $V_u \cap \mathrm{Fix}(G_N)$ is dense in $\mathrm{Fix}(G_N)$.*

Proof. Let $v \in \mathrm{Fix}(G_N)$ and take $\varepsilon > 0$. Let $T_k : \mathbb{R} \to \mathbb{R}$ be the Lipschitz function such that $T_k(s) = s$, for $|s| \leq k$, and $T_k(s) = ks|s|^{-1}$, for $|s| \geq k$. Then $T_k(v) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $T_k(v) \in \mathrm{Fix}(G_N)$. By the Lebesgue theorem, there exists $k \geq 1$ such that $\|v - T_k(v)\|_{1,p} < \varepsilon/2$. For this $k \geq 1$, for all $h \geq 1$, consider now $v_h = H(u/h)T_k(v)$, where H is defined as in the proof of Lemma 16. Then, $v_h \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v_h \in \mathrm{Fix}(G_N)$ and, of course, $u \in L^\infty(\{v_h \neq 0\})$. In turn, $v_h \in V_u \cap \mathrm{Fix}(G_N)$, $v_h(x) \to T_k(v)(x)$, $|v_h(x)| \leq |T_k(v)| \leq |v(x)|$ for a.e. $x \in \Omega$, $D_j v_h(x) \to D_j T_k(v)(x)$ a.e. in Ω as $h \to \infty$ and there exists a positive constant C (depending upon k), such that

$$\begin{aligned} |D_j v_h(x)| &= |h^{-1}H'(u/h)D_j u(x)T_k(v)(x) + H(u/h)D_j v(x)\chi_{\{|v|\leq k\}}(x)| \\ &\leq C|D_j u(x)| + C|D_j v(x)|, \quad \text{for a.e. } x \in \Omega. \end{aligned}$$

By the Lebesgue theorem, we find $h \geq 1$ so large that $\|T_k(v) - v_h\|_{1,p} < \varepsilon/2$. Then, we conclude that $\|v_h - v\|_{1,p} \leq \|v_h - T_k(v)\|_{1,p} + \|T_k(v) - v\|_{1,p} < \varepsilon$ with $v_h \in V_u \cap \mathrm{Fix}(G_N)$. □

Proposition 21. *Let $u \in \mathrm{Fix}(G_N)$. Then it holds that*

$$\begin{aligned} |df|_{\mathrm{Fix}(G_N)}|(u) \geq \sup \Bigg\{ &\int_\Omega j_t(u, |Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega j_s(u, |Du|)v \\ &+ \int_\Omega |u|^{p-2}uv - \int_\Omega |u|^{q-2}uv : v \in V_u \cap \mathrm{Fix}(G_N), \|v\|_{1,p} \leq 1 \Bigg\}. \end{aligned}$$

Moreover,

$$\int_\Omega j_t(u, |Du|)|Du| + \int_\Omega j_s(u, |Du|)u \leq |dJ|_{\mathrm{Fix}(G_N)}|(u)\|u\|_{1,p}. \tag{35}$$

In turn, if $|dJ|_{\mathrm{Fix}(G_N)}|(u) < \infty$, then $j_t(u, |Du|)|Du| \in L^1(\Omega)$ and $j_s(u, |Du|)u \in L^1(\Omega)$.

Proof. Concerning the first assertion, notice that [15, Propositions 4.5 and 6.2] extend to the case where $H_0^1(\Omega)$ is substituted by $W_0^{1,p}(\Omega)$, Ω is possibly unbounded and (21)–(23) are assumed. The assertion follows as in [15, Propositions 4.5 and 6.2] applied with f and J substituted by $f|_{\mathrm{Fix}(G_N)}$ and $J|_{\mathrm{Fix}(G_N)}$. It is enough to notice that, fixed $v \in V_u \cap \mathrm{Fix}(G_N)$, the deformation (for $\delta > 0$ and $k \geq 1$)

$$\mathscr{H}(z, t) = z + \frac{t}{1+\varepsilon}H\left(\frac{z}{k}\right)v, \quad z \in B(u, \delta), \ t \in [0, \delta],$$

that makes the job in the proof of [15, Proposition 4.5] satisfies $\mathscr{H}(z,t) \in \mathrm{Fix}(G_N)$ for every $z \in B(u,\delta) \cap \mathrm{Fix}(G_N)$. Concerning the second assertion, notice that [15, Lemma 4.6] extends to the case where $H_0^1(\Omega)$ is substituted by $W_0^{1,p}(\Omega)$, Ω is possibly unbounded and (21)–(23) hold. Furthermore, the deformation (for $\delta > 0$ and $k \geq 1$)

$$\mathscr{H}(z,t) = z - \frac{t}{\|T_k(u)\|_{1,p}(1+\varepsilon)} T_k(z), \quad z \in B(u,\delta),\ t \in [0,\delta],$$

that works inside the proof of [15, Lemma 4.6] is again such that $\mathscr{H}(z,t) \in \mathrm{Fix}(G_N)$ whenever one takes $z \in B(u,\delta) \cap \mathrm{Fix}(G_N)$. □

Proposition 22. *Let $w \in (\mathrm{Fix}(G_N))'$ and let $u \in \mathrm{Fix}(G_N)$ be such that*

$$\int_\Omega j_s(u,|Du|)v + \int_\Omega j_t(u,|Du|)\frac{Du}{|Du|} \cdot Dv + \int_\Omega |u|^{p-2}uv = \int_\Omega |u|^{q-2}uv + \langle w, v\rangle,$$

for every $v \in V_u \cap \mathrm{Fix}(G_N)$. Moreover, suppose that $j_t(u,|Du|)|Du| \in L^1(\Omega)$, and that there exist $v \in \mathrm{Fix}(G_N)$ and $\eta \in L^1(\Omega)$ such that

$$j_s(u,|Du|)v \geq \eta, \qquad j_t(u,|Du|)\frac{Du}{|Du|} \cdot Dv \geq \eta. \tag{36}$$

Then $j_s(u,|Du|)v \in L^1(\Omega)$, $j_t(u,|Du|)\frac{Du}{|Du|} \cdot Dv \in L^1(\Omega)$ and v is an admissible test.

Proof. It is a simple adaptation of [15, Theorem 4.8] to the invariant framework. □

We are now ready to provide the proof of Theorem 12.

6.0.5. Proof of Theorem 12

By Proposition 14 and Corollary 17, f satisfies condition (10) and conditions (11) and (12) of Theorem 8. Furthermore, by Lemma 15, condition (13) is satisfied as well. We now consider f restricted to $\mathrm{Fix}(G_N)$, namely

$$f|_{\mathrm{Fix}(G_N)} : \mathrm{Fix}(G_N) \to \mathbb{R} \cup \{+\infty\}.$$

Let us now prove that $f|_{\mathrm{Fix}(G_N)}$ admits a nontrivial critical point $u \in \mathrm{Fix}(G_N)$, by applying the Mountain Pass theorem for semi-continuous functionals (cf. e.g. [15, Theorem 3.9]). This result requires condition (6) to be satisfied, which holds true by Proposition 19. In light of $j(s,t) \geq \alpha_0 t^p$, there exist $\rho_0 > 0$ sufficiently small and $\sigma_0 > 0$ such that $f(u) \geq \sigma_0$ for any $u \in \mathrm{Fix}(G_N)$ with $\|u\|_{1,p} = \rho_0$. Moreover, fixed a nonzero function $\psi \in \mathrm{Fix}(G_N) \cap L^\infty(\Omega)$, by virtue of assumption (26), we can find a positive number τ such that

$$\alpha(\tau\|\psi\|_\infty)\tau^p \leq (2q)^{-1}\|\psi\|_q^q \|D\psi\|_p^{-p}\tau^q \quad \text{and} \quad \tau > \max\left\{ \left\{\frac{2q}{p}\frac{\|\psi\|_p^p}{\|\psi\|_q^q}\right\}^{\frac{1}{q-p}}, \frac{\rho_0}{\|\psi\|_{1,p}} \right\}.$$

In turn, setting $v_1 := \tau\psi \in \mathrm{Fix}(G_N)$, we obtain $\|v_1\|_{1,p} > \rho_0$ and

$$f(v_1) \leq \alpha(\tau\|\psi\|_\infty)\tau^p\|D\psi\|_p^p + \tau^p\frac{\|\psi\|_p^p}{p} - \tau^q\frac{\|\psi\|_q^q}{q} \leq \tau^p\frac{\|\psi\|_p^p}{p} - \tau^q\frac{\|\psi\|_q^q}{2q} < 0.$$

Therefore $\inf\{f(u) : u \in \mathrm{Fix}(G_N), \|u\|_{1,p} = \rho_0\} \geq \sigma_0 > 0 = \max\{f(0), f(v_1)\}$, so that f admits a positive Mountain Pass value. Furthermore, for any $c \in \mathbb{R}$, f satisfies the Palais–Smale condition at level c. Although the proof is essentially contained in [15], for the sake of self-containedness, we shall provide a sketch in the following highlighting the main differences. Then let $(u_h) \subset \mathrm{Fix}(G_N)$ with $f(u_h) \to c$ and $|df|_{\mathrm{Fix}(G_N)}|(u_h) \to 0$, as $h \to \infty$. Then, by combining Proposition 20, Proposition 21 and the Hahn–Banach theorem, there exists $(w_h) \subset (\mathrm{Fix}(G_N))'$ such that $\|w_h\|_{-1,p'} \to 0$ as $h \to \infty$, $j_t(u_h,|Du_h|)|Du_h| \in L^1(\Omega)$ (by (35), since $|df|_{\mathrm{Fix}(G_N)}|(u_h) < +\infty$ yields $|dJ|_{\mathrm{Fix}(G_N)}|(u) < +\infty$ in light of [13, Theorem 4.13(ii) combined with Theorem 5.1(ii)]) and

$$\int_\Omega j_s(u_h,|Du_h|)v + \int_\Omega j_t(u_h,|Du_h|)\frac{Du_h}{|Du_h|} \cdot Dv + \int_\Omega |u_h|^{p-2}u_h v = \int_\Omega |u_h|^{q-2}u_h v + \langle w_h, v\rangle, \tag{37}$$

for every $v \in V_{u_h} \cap \mathrm{Fix}(G_N)$. By assumption (25), the sequence $(u_h) \subset \mathrm{Fix}(G_N)$ is bounded in $\mathrm{Fix}(G_N)$ (see e.g. [24, Lemma 3.15]). Moreover, by means of Proposition 22, due to (24) we can choose the test $v = u_n$ in (37), yielding

$$\int_\Omega j_s(u_h,|Du_h|)u_h + \int_\Omega j_t(u_h,|Du_h|)|Du_h| + \int_\Omega |u_h|^p = \int_\Omega |u_h|^q + \langle w_h, u_h\rangle.$$

By the sign condition (24) and the boundedness of (u_h) in $W_0^{1,p}(\Omega)$, this yields

$$\sup_{h\geq 1}\int_\Omega j_t(u_h,|Du_h|)|Du_h|<+\infty,\qquad \sup_{h\geq 1}\int_\Omega j_s(u_h,|Du_h|)u_h<+\infty. \tag{38}$$

We claim that, up to a subsequence, (u_h) is strongly convergent to some $u\in \mathrm{Fix}(G_N)$. In this step, of course, the compact injections (18) will play a crucial rŏle. Let $u\in\mathrm{Fix}(G_N)$ be the weak limit of the sequence (u_h) in $\mathrm{Fix}(G_N)$. Since $j_t(s,t)t\geq 0$, by Fatou's Lemma the first formula in (38) yields $j_t(u,|Du|)|Du|\in L^1(\Omega)$.

By following the first part of the proof of [15, Theorem 5.1], we get $Du_h(x)\to Du(x)$ a.e. in Ω. More precisely, given a sequence of bounded G_N-invariant invading domains (Ω_k) of Ω, one is led to the application of [25, Theorem 5] to a suitable variational identity involving a class of Leray–Lions operators $b_h(x,\xi)$ related to $j(s,|\xi|)$, $(\tilde w_h)\subset(\mathrm{Fix}(G_N))'$ convergent in $\|\cdot\|_{-1,p'}$ and a sequence (f_h) bounded in $L^1(\Omega_k)$. This is achieved by choosing in (37) the admissible test functions $H(u_h/k)\tilde v\in V_{u_h}\cap\mathrm{Fix}(G_N)$, where $\tilde v\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ is the G_N-invariant function defined by setting $\tilde v(x)=0$ for a.e. $x\in\Omega\setminus\Omega_k$ and $\tilde v(x)=v(x)$ for a.e. $x\in\Omega_k$, $v\in W_0^{1,p}(\Omega_k)$ being an arbitrary bounded G_N-invariant function. In fact, after suitably rearranging the terms arising in the calculation, one is led to an identity of the form

$$\begin{cases}\displaystyle\int_{\Omega_k} b_h(x,Du_h)\cdot Dv=\langle\tilde w_h,v\rangle+\int_{\Omega_k}f_hv,\\ \text{for every } G_N\text{-invariant function } v\in W_0^{1,p}(\Omega_k)\cap L^\infty(\Omega_k).\end{cases}$$

We now point out that, in our framework, differently from [25] (cf. [25, Formula 19]) the above variational identity only holds for G_N-invariant test functions. On the other hand, since Ω_k and u_h are G_N-invariant, this is actually sufficient to succeed. In fact, for a suitable sequence of numbers $\delta_k>0$, the only test functions which arise in the proof of [25, Theorem 5] are indeed G_N-invariant functions, being exactly of the form

$$v_h:=\varphi_K\,\psi_{\delta_h}\circ(u_h-u)\in W_0^{1,p}(\Omega_k)\cap L^\infty(\Omega_k),$$

where K is a fixed G_N-invariant compact subset of Ω_k, $\varphi_K\in C_c^\infty(\Omega_k)$ with $0\leq\varphi_K\leq 1$ is a G_N-invariant function with $\varphi_K=1$ on K and $\mathrm{supt}(\varphi_K)\subseteq U\subseteq\bar U\subseteq\Omega_k$ for some G_N-invariant open set $U\subseteq\Omega_k$ and $\psi_\delta\in C_c^1(\mathbb{R})$, $\psi_\delta(s)=s$ for $\{|s|\leq\delta\}$, $\psi_\delta(s)=0$ for $\{|s|\geq 2\delta\}$, $|\psi_\delta|\leq 2\delta$ and $|D\psi_\delta|$ is bounded. Due to the G_N-invariance of Ω_k restricting to G_N-invariant compacts K of Ω_k is without loss of generality. In conclusion, [25, Theorem 5] allows us to conclude that $Du_h(x)\to Du(x)$ for a.e. x in the set $E_k=\{x\in\Omega_k:|u(x)|\leq k\}$. Finally, the property on Ω follows by a standard diagonal argument on the E_ks. Moreover, taking suitable test functions in $\mathrm{Fix}(G_N)$ (cf. [15, test (5.8) and the test above (5.11)]), we get

$$\int_\Omega j_s(u,|Du|)v+\int_\Omega j_t(u,|Du|)\frac{Du}{|Du|}\cdot Dv+\int_\Omega|u|^{p-2}uv=\int_\Omega|u|^{q-2}uv, \tag{39}$$

for every $v\in V_u\cap\mathrm{Fix}(G_N)$. For $M>0$, consider the Lipschitz function $\zeta:\mathbb{R}\to\mathbb{R}^+$,

$$\zeta(s)=M|s|\quad\text{for } |s|\leq R,\qquad \zeta(s)=MR\quad\text{for } |s|\geq R. \tag{40}$$

By (24), the growth condition on j_s and $j_t(s,t)t\geq\alpha_0t^p$, we can easily choose M (depending upon R and α_0) such that $[j_s(s,t)+j_t(s,t)t\zeta'(s)]s\geq 0$ for all $s\in\mathbb{R}$ and $t\in\mathbb{R}^+$. Recalling that $j_t(u_h,|Du_h|)|Du_h|\in L^1(\Omega)$ and $j_t(u,|Du|)|Du|\in L^1(\Omega)$, by Proposition 22, condition (24) and the definition of ζ, the function $v_h:=u_he^{\zeta(u_h)}\in\mathrm{Fix}(G_N)$ (resp. $v:=ue^{\zeta(u)}\in\mathrm{Fix}(G_N)$) can be taken as admissible test functions into (37) (resp. in (39)). Finally, since by virtue of Lemma 11 (u_h) strongly converges to u in $L^q(\Omega)$, by dominated convergence,

$$\lim_h\int_\Omega|u_h|^qe^{\zeta(u_h)}=\int_\Omega|u|^qe^{\zeta(u)}.$$

Therefore, using Fatou's Lemma twice, we reach

$$\begin{aligned}\int_\Omega j_t(u,|Du|)|Du|e^{\zeta(u)}+\int_\Omega|u|^pe^{\zeta(u)}&\leq\liminf_h\left[\int_\Omega j_t(u_h,|Du_h|)|Du_h|e^{\zeta(u_h)}+\int_\Omega|u_h|^pe^{\zeta(u_h)}\right]\\&\leq\limsup_h\left[\int_\Omega j_t(u_h,|Du_h|)|Du_h|e^{\zeta(u_h)}+\int_\Omega|u_h|^pe^{\zeta(u_h)}\right]\\&=\int_\Omega|u|^qe^{\zeta(u)}-\liminf_h\int_\Omega\left[j_s(u_h,|Du_h|)+j_t(u_h,|Du_h|)|Du_h|\zeta'(u_h)\right]u_he^{\zeta(u_h)}\\&\leq\int_\Omega|u|^qe^{\zeta(u)}-\int_\Omega\left[j_s(u,Du)+j_t(u,|Du|)|Du|\zeta'(u)\right]ue^{\zeta(u)}\\&=\int_\Omega j_t(u,|Du|)|Du|e^{\zeta(u)}+\int_\Omega|u|^pe^{\zeta(u)},\end{aligned}$$

which combined with the pointwise convergence of the gradients and the inequalities

$$\alpha_0|Du_h|^p \le j_t(u_h, |Du_h|)|Du_h|e^{\zeta(u_h)} + |u_h|^p e^{\zeta(u_h)}$$
$$|u_h|^p \le j_t(u_h, |Du_h|)|Du_h|e^{\zeta(u_h)} + |u_h|^p e^{\zeta(u_h)}$$

implies that $\|u_h\|_p \to \|u\|_p$ and $\|Du_h\|_p \to \|Du\|_p$ as $h \to \infty$, by dominated convergence. By the uniform convexity of L^p, we conclude that $u_h \to u$ in $W_0^{1,p}(\Omega)$ as $h \to \infty$. The Palais–Smale condition is thus verified. By applying [15, Theorem 3.9], we find a nontrivial Mountain Pass critical point $u \in \mathrm{Fix}(G_N)$, namely $|df|_{\mathrm{Fix}(G_N)}|(u) = 0$. In turn, in light of Proposition 2, $0 \in \partial f|_{\mathrm{Fix}(G_N)}(u)$. Hence, by Theorem 8 (more precisely by Corollary 18), we get

$$\forall v \in V_u: \quad f'(u)v = 0.$$

Now, by Proposition 14, we have

$$\forall v \in V_u: \quad \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|}\cdot Dv + \int_\Omega j_s(u, |Du|)v + \int_\Omega |u|^{p-2}uv = \int_\Omega |u|^{q-2}uv.$$

In light of Proposition 21, it follows that $j_t(u, |Du|)|Du| \in L^1(\Omega)$ and $j_s(u, |Du|)u \in L^1(\Omega)$. Then, it holds that $|j_s(u, |Du|)| \le |j_s(u, |Du|)|\chi_{\{|u|\le 1\}} + |j_s(u, |Du|)u|$. Also, for a fixed compact $K \subset \Omega$, it holds that

$$\begin{aligned}|j_t(u, |Du|)|\chi_K &\le |j_t(u, |Du|)\chi_{\{|Du|\ge 1\}}\chi_K| + |j_t(u, |Du|)\chi_{\{|Du|\le 1\}}\chi_K| \\ &\le j_t(u, |Du|)|Du| + C\alpha(|u|)\chi_K \\ &\le j_t(u, |Du|)|Du| + C|u|^{p^*} + C\chi_K.\end{aligned}$$

We have thus proved that $j_s(u, |Du|) \in L^1(\Omega)$ and $j_t(u, |Du|) \in L^1_{\mathrm{loc}}(\Omega)$. In turn, for any $v \in C_c^\infty(\Omega)$, we have $j_s(u, |Du|)v \in L^1(\Omega)$ and $j_t(u, |Du|)\frac{Du}{|Du|}\cdot Dv \in L^1(\Omega)$, so that, in light of [15, Theorem 4.8] (which extends to the current setting), we get

$$\forall v \in C_c^\infty(\Omega): \quad \int_\Omega j_t(u, |Du|)\frac{Du}{|Du|}\cdot Dv + \int_\Omega j_s(u, |Du|)v + \int_\Omega |u|^{p-2}uv = \int_\Omega |u|^{q-2}uv,$$

namely u is a distributional solution. The proof is now complete. □

6.0.6. *Proof of Corollary 13*

Let $\Omega = \mathbb{R}^N$ and take $X = W^{1,p}(\mathbb{R}^N)$ and $G = \mathcal{O}(N)$. Then, the assertion follows directly from Theorem 12 since the invariance $u(gx) = u(x)$ for all $g \in G$ is equivalent to the radial symmetry of u. To obtain that u is nonnegative, in the application of Theorem 12 it suffices to replace, in the definition of f, the term $\int_{\mathbb{R}^N} |u|^q/q$ with $\int_{\mathbb{R}^N} G(u)$, where $G(s) = (s^+)^q/q$. Hence, as in the proof of Theorem 12, after the application of our second Palais symmetric criticality principle, one finds a $u \in \mathrm{Fix}(G)$ which solves

$$\int_{\mathbb{R}^N} j_t(u, |Du|)\frac{Du}{|Du|}\cdot Dv + \int_{\mathbb{R}^N} j_s(u, |Du|)v + \int_{\mathbb{R}^N} |u|^{p-2}uv = \int_{\mathbb{R}^N} |u^+|^{q-2}uv,$$

for all $v \in V_u$. If ζ is the map defined in (40), the function $\tilde{v} = -u^- e^{\zeta(u)}$ can be taken as an admissible test function in the above identity by [15, Theorem 4.8] in view of the growth assumptions on j_s, j_t and of the sign condition (24), yielding

$$\begin{aligned}0 &= \int_{\mathbb{R}^N} |u^+|^{q-2}u\tilde{v} = \int_{\mathbb{R}^N} e^{\zeta(u)} j_t(u, |Du^-|)|Du^-| \\ &\quad + \int_{\mathbb{R}^N} -[j_s(u, |Du|) + j_t(u, |Du|)|Du|\zeta'(u)]u^- e^{\zeta(u)} + \int_{\mathbb{R}^N} |u^-|^p e^{\zeta(u)} \\ &\ge \int_{\mathbb{R}^N} |u^-|^p e^{\zeta(u)} \ge \int_{\mathbb{R}^N} |u^-|^p \ge 0,\end{aligned}$$

yielding $u^- = 0$, hence $u \ge 0$. This concludes the proof. □

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