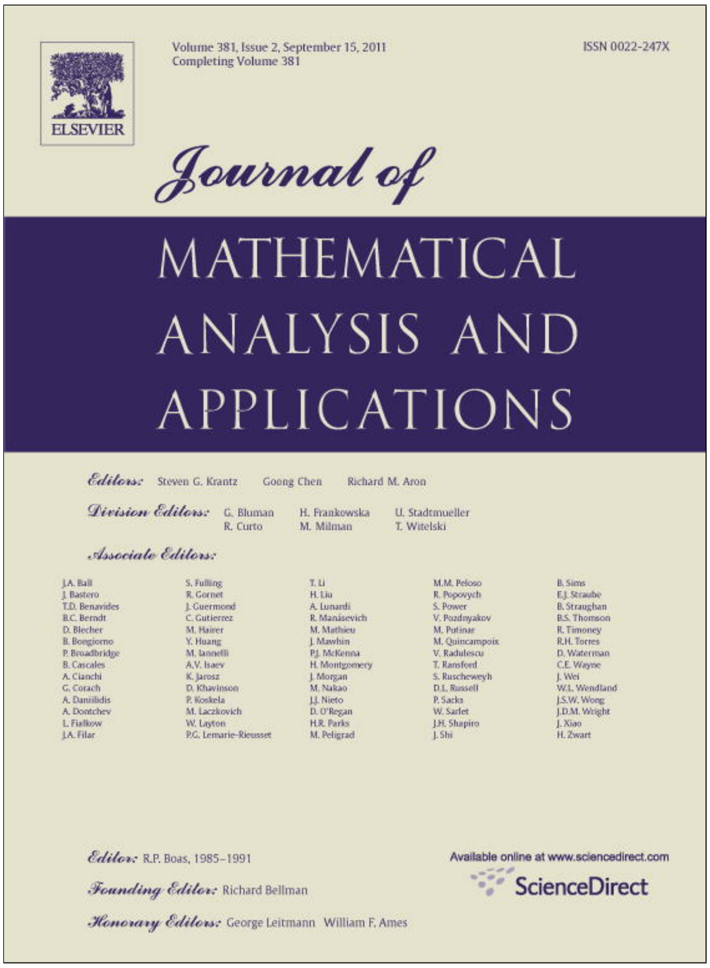
Volume 381, Issue 2, September 15, 2011
Completing Volume 381
ISSN 0022-247X
ELSEVIER
Journal of
MATHEMATICAL ANALYSIS AND APPLICATIONS
Editors: Steven G. Krantz Goong Chen Richard M. Aron
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Available online at www.sciencedirect.com
ScienceDirect

J. Math. Anal. Appl. 381 (2011) 857–865

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mountain Pass solutions for quasi-linear equations via a monotonicity trick

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ARTICLE INFO

Article history:
Received 1 March 2011
Available online 5 April 2011
Submitted by P.J. McKenna

Keywords:
Non-smooth critical point theory
Monotonicity trick
Palais–Smale condition

ABSTRACT

We obtain the existence of symmetric Mountain Pass solutions for quasi-linear equations without the typical assumptions which guarantee the boundedness of an arbitrary Palais–Smale sequence. This is done through a recent version of the monotonicity trick proved in Squassina (in press) [22]. The main results are new also for the p-Laplacian operator.

1. Introduction

Let $N > p > 1$. In the study of the quasi-linear partial differential equation

$$-\operatorname{div}\big(j_\xi(u, Du)\big) + j_s(u, Du) + V(x)|u|^{p-2}u = g(u), \quad u \in W^{1,p}\big(\mathbb{R}^N\big) \tag{1}$$

by means of variational methods, a rather typical assumption on $j(s,\xi)$ and $g(s)$ is that there exist $p < q < Np/(N-p)$ and $\delta > 0$ such that

$$qj(s,\xi) - j_s(s,\xi)s - (1+\delta)j_\xi(s,\xi)\cdot\xi - qG(s) + g(s)s \geqslant 0, \tag{2}$$

for all $s \in \mathbb{R}$ and any $\xi \in \mathbb{R}^N$ (cf. [2,7]). This condition ensures that *every* Palais–Smale sequence, in a suitable sense, of the associated functional $f : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$,

$$f(u) = \int_{\mathbb{R}^N} j(u, Du) + \frac{1}{p}\int_{\mathbb{R}^N} V(x)|u|^p - \int_{\mathbb{R}^N} G(u),$$

is *bounded* in $W^{1,p}(\mathbb{R}^N)$. We might refer to this technical condition as the generalized Ambrosetti–Rabinowitz condition, involving the terms of the quasi-linear operator j. In fact, in the treatment of the non-autonomous semi-linear equation

$$-\Delta u + V(x)u = g(u), \quad u \in H^1\big(\mathbb{R}^N\big), \tag{3}$$

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1 Author partially supported by MIUR Project: *Metodi Variazionali e Topologici nello Studio di Fenomeni non Lineari.*

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.04.014

the previous inequality (2) reduces to the classical Ambrosetti–Rabinowitz condition [1], namely $0 < qG(s) \leqslant g(s)s$, for every $s \in \mathbb{R}$. Of course, aiming to achieve the existence of *multiple* solutions for Eq. (1), one needs to know that the Palais–Smale condition for f is satisfied at an *arbitrary* energy level, and hence it is necessary to guarantee that Palais–Smale sequences are always at least bounded, through condition (2). On the contrary, under suitable assumptions, if one merely focuses on the existence of a nonnegative Mountain Pass solution of (1), it is reasonable to expect that by a clever selection of a *special* Palais–Smale sequence at the Mountain Pass level c one could reach the goal of getting a solution to (1) without knowing that the Palais–Smale condition holds. The existence of such a nice sequence is possible since the definition of c allows to detect continuous paths $\gamma : [0,1] \to W^{1,p}(\mathbb{R}^N)$ with a very good behavior. The idea, considering for instance problems (3), is to see $f = f_1$ as the end point of the continuous family of C^1 functionals $f_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$f_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 - \lambda \int_{\mathbb{R}^N} G(u).$$

When f_λ satisfies a uniform Mountain Pass geometry, then it is possible to use the so-called monotonicity trick for C^1 smooth functionals, originally discovered by Struwe [24] in a very special setting and generalized and formalized later in an abstract framework by Jeanjean [12] and Jeanjean and Toland [15]. This strategy provides a bounded Palais–Smale sequence for all λ fixed, up to a set of null measure. Then, by requiring some compactness condition one can detect a sequence (λ_j), increasingly converging to 1, for which there corresponds a sequence (u_{λ_j}) of solutions to (3) at the Mountain Pass level c_{λ_j}, namely

$$c_\lambda = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} f_\lambda\big(\gamma(t)\big), \qquad \Gamma = \big\{\gamma \in C\big([0,1], W^{1,p}\big(\mathbb{R}^N\big)\big):\ \gamma(0) = 0,\ \gamma(1) = w\big\}, \tag{4}$$

$w \in W^{1,p}(\mathbb{R}^N)$ being a suitable function with $f_\lambda(w) < 0$ for any value of λ. Then, u_{λ_j} being exact solutions, one can exploit the Pohǒzaev identity and combine it with the energy level constraint to show in turn that (u_{λ_j}) is a *bounded* Palais–Smale sequence for f_1. In the case of *semi-linear* equations such as (3), we refer the reader to [14,3] where the approach has been successfully developed. The main goal of this manuscript is twofold. On one hand, we intend to show how condition (2) can be completely removed by using a general version of the monotonicity trick recently developed in [22] in the framework of the non-smooth critical point theory of [9,8]. In this respect, first, in order to analyze the most clarifying concrete situation, we consider a class of functionals invariant under orthogonal transformations, set in the space of radial functions (see Theorem 1). As in the smooth case, by studying a penalized functional f_λ we will obtain a sequence of λ_j converging to one, with corresponding weak solutions u_{λ_j}. In order to obtain that the sequence (u_{λ_j}) is bounded, a general version of the Pohǒzaev identity [10] for merely C^1 weak solutions will be crucial, as $C^{1,\alpha}$ is the optimal regularity if $p \neq 2$ [25]. Moreover, a generalized version of Palais' symmetric criticality principle recently achieved in [21] will be exploited. These results are new also in the particular meaningful case $j(u, Du) = |Du|^p/p$ with $p \neq 2$, being the case $p = 2$ covered in [3]. On the other hand, when one does not restrict the functional to the space of radially symmetric functions (see Theorem 2), it is possible to make a stronger use of the result in [22] to construct a bounded, almost symmetric (cf. (27)), Palais–Smale sequence which will give a radial and radially decreasing solution. At the high level of generality of Eq. (1), proving a priori that the radial solution is decreasing seems a particularly strong fact. These results are new also for $j(u, Du) = |Du|^p/p$, even with $p = 2$.

Let us now state the main results of the paper. Let $N > p > 1$ and let $j : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 function such that the map $t \mapsto j(s,t)$ is increasing and strictly convex. Moreover, we assume that there exist $\alpha, \beta > 0$ with

$$\alpha t^p \leqslant j(s,t) \leqslant \beta t^p, \quad \text{for every } s \in \mathbb{R} \text{ and } t \in \mathbb{R}^+, \tag{5}$$

$$\big|j_s(s,t)\big| \leqslant \beta t^p, \qquad \big|j_t(s,t)\big| \leqslant \beta t^{p-1}, \quad \text{for every } s \in \mathbb{R} \text{ and } t \in \mathbb{R}^+, \tag{6}$$

$$j_s(s,t)s \geqslant 0, \quad \text{for every } s \in \mathbb{R} \text{ and } t \in \mathbb{R}^+. \tag{7}$$

Let $V : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 function such that there exist $m, M \in \mathbb{R}^+$ with

$$0 < m \leqslant V(\tau) \leqslant M, \quad \text{for every } \tau \in \mathbb{R}^+. \tag{8}$$

Furthermore, we shall assume that

$$\big\|V'\big(|x|\big)|x|\big\|_{L^{N/p}(\mathbb{R}^N)} < \alpha p \mathcal{S}, \tag{9}$$

where $\mathcal{S} = \inf\{\|Du\|_p^p:\ u \in W^{1,p}(\mathbb{R}^N),\ \|u\|_{L^{Np/(N-p)}(\mathbb{R}^N)} = 1\}$ is the best Sobolev constant and α is the number in (5). Apart from the natural growths (5)–(6), condition (7) is a typical requirement in the frame of quasi-linear equations, which helps [2,7,18,20,23] in the achievement of both existence and summability issues related to Eq. (1). Under (5) and (8), the functional defined either in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ or in $W^{1,p}(\mathbb{R}^N)$ as

$$u \mapsto \int_{\mathbb{R}^N} j\big(u, |Du|\big) + V\big(|x|\big)\frac{|u|^p}{p},$$

is continuous but not even locally Lipschitz, as it can be easily checked. Moreover, it admits Gateaux derivatives along any bounded direction v, but not on an arbitrary direction v of either $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ or $W^{1,p}(\mathbb{R}^N)$. This is the reason why we will make use of the abstract machinery developed in [9,8] for continuous functionals, the related monotonicity trick proved in [22] and Palais' symmetric criticality principle formulated in [21].

Let $p^* := Np/(N-p)$ and consider the equation

$$-\operatorname{div}\left[j_t\big(u,|Du|\big)\frac{Du}{|Du|}\right]+j_s\big(u,|Du|\big)+V\big(|x|\big)u^{p-1}=g(u)\quad \text{in } \mathbb{R}^N. \tag{10}$$

Our first main result is the following

Theorem 1. *Assume* (5)–(9) *and let $g:\mathbb{R}^+\to\mathbb{R}^+$ be continuous with $g(0)=0$ and extended by zero on $\mathbb{R}^-$. Moreover,*

$$\lim_{s\to 0^+}\frac{g(s)}{s^{p-1}}=\lim_{s\to+\infty}\frac{g(s)}{s^{p^*-1}}=0, \tag{11}$$

and, furthermore, for $G(s)=\int_0^s g(t)$,

$$\text{there exists } s>0 \text{ such that } pG(s)-Ms^p>0. \tag{12}$$

Then Eq. (10) *admits a nontrivial, nonnegative, distributional and radially symmetric solution $u\in W^{1,p}(\mathbb{R}^N)$.*

This result seems new even in the particular p-Laplacian case $j(s,t)=t^p/p$ with $p\neq 2$. In order to prove Theorem 1, we consider the continuous functionals $f_\lambda : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)\to\mathbb{R}$,

$$f_\lambda(u)=\int_{\mathbb{R}^N} j\big(u,|Du|\big)+\int_{\mathbb{R}^N} V\big(|x|\big)\frac{|u|^p}{p}-\lambda\int_{\mathbb{R}^N} G(u),\quad \lambda\in[\delta,1], \tag{13}$$

for some suitable value of $\delta\in(0,1)$. First we shall prove that f_λ fulfills a uniform Mountain Pass geometry. Next we show that for all $\lambda\in(\delta,1]$ any bounded Palais–Smale sequence is, actually, strongly convergent. Furthermore, by applying the monotonicity trick of [22] and Palais' symmetric criticality principle proved in [21] for continuous functionals, a sequence $\lambda_h\subset[\delta,1)$ with $\lambda_h\nearrow 1$ is detected such that, for each $h\geqslant 1$, there exists a distributional solution $u_{\lambda_h}\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ of

$$-\operatorname{div}\left[j_t\big(u,|Du|\big)\frac{Du}{|Du|}\right]+j_s\big(u,|Du|\big)+V\big(|x|\big)u^{p-1}=\lambda_h g(u)\quad \text{in } \mathbb{R}^N$$

at the Mountain Pass level c_{λ_h}. Then, by exploiting a Pohǒzaev identity [10] for C^1 solutions of (10), we show in turn that (u_{λ_h}) is also a bounded Palais–Smale sequence for f_1, and passing to the limit will provide the desired conclusion.

Our second main result is the following

Theorem 2. *Assume* (5)–(9)*, let $g:\mathbb{R}^+\to\mathbb{R}^+$ be continuous with $g(0)=0$, extended by zero on $\mathbb{R}^-$, satisfying* (12)*, and such that for all $\varepsilon>0$ there is $C_\varepsilon\in\mathbb{R}^+$ with*

$$\big|g(s)\big|\leqslant\varepsilon s^{p-1}+C_\varepsilon s^{q-1},\quad p<q<p^*, \tag{14}$$

for every $s\in\mathbb{R}^+$. Let V also satisfy

$$|x|\leqslant|y|\quad\Longrightarrow\quad V\big(|x|\big)\leqslant V\big(|y|\big)\quad \text{for every } x,y\in\mathbb{R}^N. \tag{15}$$

Then Eq. (10) *admits a nontrivial, nonnegative, distributional, radially symmetric and decreasing solution $u\in W^{1,p}(\mathbb{R}^N)$.*

This result seems new even in the particular p-Laplacian case $j(s,t)=t^p/p$, included $p=2$ due to the monotonicity information which is obtained a priori, skipping a posteriori PDEs arguments. In place of (11), here we need the slightly more restrictive condition (14), since we cannot work directly on sequences of radial functions, which enjoy uniform decay properties. In order to prove Theorem 2, we argue on the continuous functionals $f_\lambda : W^{1,p}(\mathbb{R}^N)\to\mathbb{R}$ again defined as in (13) for all $\lambda\in(\delta,1]$, for a suitable $\delta\in(0,1)$. Hence here we do not restrict the functional to the space of radially symmetric functions. However, we still proceed as indicated above for the proof Theorem 1, but, by exploiting the symmetry properties of the functional under polarization (cf. [22]) we use the symmetry features of the monotonicity trick of [22] and we obtain the existence of a bounded and almost symmetric (cf. (27)) Palais–Smale sequence for f_1. Possessing a compactness result for such sequences, we can conclude the proof. We remark that in this second statement the solution found is not only radially symmetric, but also automatically *radially decreasing*.

In both Theorems 1 and 2, the radial dependence of the potential V is crucial in order to detect suitable precompact Palais–Smale sequences, while in the general case an accurate description of the behavior of such sequences is required. This analysis was carried on in [14] for $j(s,t) = t^2/2$ using concentration compactness arguments [17] which, to our knowledge, are not yet available for a general j.

In the autonomous case, namely the case where the potential V is constant, based upon scaling arguments, other classical approaches can be adopted also allowing more general classes of nonlinearities g. We refer the reader to Berestycki and Lions [4] for the semi-linear case and to [5,13] for more general situations including quasi-linear equations under homogeneity assumptions on j which are not assumed here.

Finally we remark that, while in Theorem 1 the solution is found at the *restricted* Mountain Pass level

$$c_{\mathrm{rad}} = \inf_{\gamma\in\Gamma_{\mathrm{rad}}} \sup_{t\in[0,1]} f_1\big(\gamma(t)\big), \qquad \Gamma_{\mathrm{rad}} = \big\{\gamma \in C\big([0,1], W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)\big):\ \gamma(0)=0,\ \gamma(1)=w\big\},$$

in Theorem 2 the solution is found at the *global* Mountain Pass level

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} f_1\big(\gamma(t)\big), \qquad \Gamma = \big\{\gamma \in C\big([0,1], W^{1,p}\big(\mathbb{R}^N\big)\big):\ \gamma(0)=0,\ \gamma(1)=w\big\}.$$

Of course, on one hand, we have $c \leqslant c_{\mathrm{rad}}$. On the other hand it is not clear if, in general, one has $c = c_{\mathrm{rad}}$ or $c < c_{\mathrm{rad}}$ although, precisely as a *further consequence* of Theorem 2, this occurs when V is constant and the map $t \mapsto j(s,t)$ is p-homogeneous (see Remark 1).

2. Proof of Theorem 1

We will prove Theorem 1 by studying the functionals $f_\lambda : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ defined in (13). Taking into account assumptions (5), (8) and (11), recalling [4, Theorem A.VI], it follows that f_λ is well defined and (merely) continuous. In turn, we shall exploit the non-smooth critical point theory of [9,8] including the connection between critical points in a suitable sense and solutions of the associated Euler's equation (see for instance [18, Theorem 3] and also [21, Proposition 6.16] for the symmetric setting). More precisely under assumption (5)–(9), the critical points of f_λ are distributional solutions of

$$-\operatorname{div}\left[j_t\big(u,|Du|\big)\frac{Du}{|Du|}\right] + j_s\big(u,|Du|\big) + V\big(|x|\big)|u|^{p-2}u = \lambda g(u) \quad \text{in } \mathbb{R}^N. \tag{16}$$

Combining the following two lemmas shows that the minimax class (4) is nonempty and that the family (f_λ) enjoys a uniform Mountain Pass geometry whenever λ varies inside the interval $[\delta_0, 1]$, for a suitable $\delta_0 > 0$.

Lemma 3. *Assume* (5), (8) *and* (11)–(12). *Then there exists $\delta_0 \in (0,1)$ and a curve $\gamma \in C([0,1], W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))$, independent of λ, such that $f_\lambda(\gamma(1)) < 0$, for every $\lambda \in [\delta_0, 1]$.*

Proof. Due to (12), there exists $z \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $z \geqslant 0$ and Schwarz symmetric, such that

$$\int_{\mathbb{R}^N} \left(G(z) - \frac{M}{p} z^p\right) > 0.$$

To see this, follow closely the first part of [4, Step 1, pp. 324–325]. In turn, let $\delta_0 \in (0,1)$ with

$$\int_{\mathbb{R}^N} \left(\delta_0 G(z) - \frac{M}{p} z^p\right) > 0, \tag{17}$$

and define the curve $\eta \in C([0,\infty), W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))$ by setting $\eta(t) := z(\cdot/t)$ for $t \in (0,\infty)$ and $\eta(0) := 0$. From (5) and (8) it follows that

$$f_\lambda\big(\eta(t)\big) \leqslant \beta t^{N-p} \|Dz\|^p_{L^p(\mathbb{R}^N)} - t^N \int_{\mathbb{R}^N} \left(\delta_0 G(z) - \frac{M}{p} z^p\right),$$

yielding, on account of (17), a time $t_0 > 0$ such that $f_\lambda(\eta(t_0)) < 0$ for every $\lambda \in [\delta_0, 1]$. Then, the curve $\gamma \in C([0,1], W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))$, independent of λ, defined by $\gamma(t) := \eta(t_0 t)$ has the required property and Γ is nonempty by taking $w := \gamma(1)$. $\square$

Lemma 4. *Assume* (5), (8) *and* (11). *Let $\delta_0 > 0$ be the number found in Lemma* 3. *There exist $\sigma > 0$ and $\rho > 0$, independent of λ, such that $f_\lambda(u) \geqslant \sigma$ for any u in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ with $\|u\|_{1,p} = \rho$ and for every $\lambda \in [\delta_0, 1]$.*

Proof. Condition (11) implies that for every $\varepsilon > 0$, there exists C_ε such that

$$\left|g(s)\right| \leqslant \varepsilon s^{p-1} + C_\varepsilon s^{p^*-1}, \quad \text{for every } s \in \mathbb{R}^+. \tag{18}$$

Then, fixed $\varepsilon_0 < m$, we find C_{ε_0} such that for every $\lambda \in [\delta_0, 1]$

$$f_\lambda(u) \geqslant \alpha \|Du\|^p_{L^p(\mathbb{R}^N)} + \frac{m-\varepsilon_0}{p}\|u\|^p_{L^p(\mathbb{R}^N)} - C_{\varepsilon_0}\|u\|^{p^*}_{W^{1,p}(\mathbb{R}^N)}.$$

This last inequality immediately gives the conclusion. □

We will use the following compactness condition.

Definition 1. Let $\lambda, c \in \mathbb{R}$. We say that f_λ satisfies the concrete-$(BPS)_c$ condition if any bounded sequence $(u_h) \subset W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that there is $w_h \in W^{-1,p'}_{\text{rad}}(\mathbb{R}^N)$ with

$$f_\lambda(u_h) \to c, \qquad \left\langle f'_\lambda(u_h), v\right\rangle = \langle w_h, v\rangle \quad \text{for every } v \in C^\infty_{c,\text{rad}}\left(\mathbb{R}^N\right), \text{ and } w_h \to 0 \tag{19}$$

admits a strongly convergent subsequence.

In the next result we will use the property

$$j_t(s,t)t \geqslant \alpha t^p, \tag{20}$$

which can be obtained by hypotheses (5) once one has observed that, as j is a strict convex function with respect to t, it results $0 = j(s,0) \geqslant j(s,t) + j_t(s,t)\cdot(0-t)$.

Proposition 5. *Let $\lambda \in [\delta_0, 1]$, $c \in \mathbb{R}$ and assume* (5)–(8) *and* (11)*. Then the functional f_λ satisfies the concrete-$(BPS)_c$.*

Proof. Let $(u_h) \subset W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ be a bounded sequence which satisfies the properties in (19). Then, in turn, there exists a subsequence, still denoted by (u_h), converging weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, strongly in $L^q(\mathbb{R}^N)$ for any $q \in (p, p^*)$ and almost everywhere to a function $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Moreover, we can apply the result in [6] to obtain that Du_h converges to Du almost everywhere. More precisely, since the variational formulation is here restricted to radial functions, this property follows by arguing as in [21, proof of Theorem 6.4]. Then, it is possible to follow the same arguments used in [18, Step 2 of Lemma 2] (see also [20]) for bounded domains, in order to pass to the limit in the equation in (19) and obtain in turn that u satisfies the variational identity

$$\int_{\mathbb{R}^N} j_t\left(u,|Du|\right)\frac{Du}{|Du|}\cdot D\varphi + \int_{\mathbb{R}^N} j_s\left(u,|Du|\right)\varphi + \int_{\mathbb{R}^N} V\left(|x|\right)|u|^{p-2}uv = \lambda\int_{\mathbb{R}^N} g(u)\varphi, \quad \forall\varphi \in C^\infty_{c,\text{rad}}\left(\mathbb{R}^N\right).$$

In fact, all the particular test functions built in [18,20] to achieve this identity are radial, since each u_h is radial and φ is a fixed radial function. Observe also that a function $\varphi \in W^{1,p}_{\text{rad}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ can be approximated, in the $\|\cdot\|_{1,p}$ norm, by a sequence $(\varphi_m) \subset C^\infty_{c,\text{rad}}(\mathbb{R}^N)$ with $\|\varphi_m\|_{L^\infty} \leqslant c(\varphi)$, for some positive constant $c(\varphi)$. Whence, exploiting (6)–(8) and (11), recalling that u is radial and arguing as in [18, Proposition 1], it follows that u is an admissible test function, namely

$$\int_{\mathbb{R}^N} j_t\left(u,|Du|\right)|Du| + \int_{\mathbb{R}^N} j_s\left(u,|Du|\right)u + \int_{\mathbb{R}^N} V\left(|x|\right)|u|^p = \lambda\int_{\mathbb{R}^N} g(u)u. \tag{21}$$

Furthermore, taking into account that $u_h \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ and exploiting conditions (11), we can use [4, Theorem A.I] to obtain that

$$\lim_{h\to\infty}\int_{\mathbb{R}^N} g(u_h)u_h = \int_{\mathbb{R}^N} g(u)u.$$

Observe that, applying Fatou's Lemma in view of (7)–(8) and (20), formula (21) implies

$$\begin{aligned}\int_{\mathbb{R}^N} j_t\left(u,|Du|\right)|Du| + V\left(|x|\right)|u|^p &\leqslant \liminf_{h\to\infty}\left\{\int_{\mathbb{R}^N} j_t\left(u_h,|Du_h|\right)|Du_h| + V\left(|x|\right)|u_h|^p\right\}\\ &\leqslant \limsup_{h\to\infty}\left\{\int_{\mathbb{R}^N} j_t\left(u_h,|Du_h|\right)|Du_h| + V\left(|x|\right)|u_h|^p\right\}\end{aligned}$$

$$\leqslant -\liminf_{h\to\infty}\int_{\mathbb{R}^N} j_s\big(u_h,|Du_h|\big)u_h+\lim_{h\to\infty}\lambda\int_{\mathbb{R}^N} g(u_h)u_h$$
$$=-\int_{\mathbb{R}^N} j_s\big(u,|Du|\big)u+\lambda\int_{\mathbb{R}^N} g(u)u$$
$$=\int_{\mathbb{R}^N} j_t\big(u,|Du|\big)|Du|+V\big(|x|\big)|u|^p.$$

Then, taking into account (8) and (20), it results

$$\lim_{h\to\infty}\int_{\mathbb{R}^N}|Du_h|^p+m|u_h|^p=\int_{\mathbb{R}^N}|Du|^p+m|u|^p,$$

giving the desired convergence of (u_h) to u via the uniform convexity of $W^{1,p}(\mathbb{R}^N)$. □

Next, we state the main technical tool for the proof of the first theorem.

Lemma 6. *Assume that conditions* (5)–(8) *and* (11)–(12) *hold and that f_λ satisfies the concrete-(BPS)$_c$ for all $c\in\mathbb{R}$ and all $\lambda\in[\delta_0,1]$. Then there exists a sequence $(\lambda_j,u_j)\subset[\delta_0,1]\times W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ with $\lambda_j\nearrow 1$ and where u_j is a distributional solution to*

$$-\operatorname{div}\left[j_t\big(u,|Du|\big)\frac{Du}{|Du|}\right]+j_s\big(u,|Du|\big)+V\big(|x|\big)|u|^{p-2}u=\lambda_j g(u)\quad \text{in } \mathbb{R}^N, \tag{22}$$

such that $f_{\lambda_j}(u_j)=c_{\lambda_j}$.

Proof. The result follows by applying [22, Corollary 3.3] to the minimax class defined in (4), with the choice of spaces $X=S=V=W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and by defining $u^H:=u$ and $u^*:=u$ as the identity maps. In fact, assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ are fulfilled thanks to Lemmas 3 and 4. Condition $(\mathcal{H}_3)$ is implied by the structure of f_λ as it can be verified by a straightforward direct computation. Finally assumption $(\mathcal{H}_4)$ is evidently satisfied since u^H is the identity map. Since $X=W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, it turns out that, a priori, the solutions (u_j) provided by [22, Corollary 3.3] are distributional with respect to test functions in $C^\infty_{c,\mathrm{rad}}(\mathbb{R}^N)$. The fact that u_j is, actually, a distributional solution with respect to any test function in $C^\infty_c(\mathbb{R}^N)$ follows by [21, Theorem 4.1 and end of the proof of Theorem 6.4]. □

Proposition 7. *Assume* (5), (8) *and* (11)–(12)*. The map $\lambda\to c_\lambda$ is non-increasing and continuous from the left.*

Proof. The fact that c_λ is non-increasing trivially follows from the fact that $G\geqslant 0$. The proof of the left-continuity follows arguing by contradiction exactly as done in [12, Lemma 2.3]. □

2.1. Proof of Theorem 1 concluded

Proposition 5 allows us to apply Lemma 6 and obtain, in turn, a sequence u_j of distributional solution of (22) at the energy level c_{λ_j}. Following the argument in [11, Lemma 4.1] and applying [19, Theorem 1 and Remark, p. 261] one obtains $u_j\in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and then, via standard regularity arguments (see [16]) $u_j\in C^{1,\alpha}(\mathbb{R}^N)$. As a consequence, we can apply the Pohǒzaev variational identity for C^1 solutions of Eq. (22) stated in [10, Lemma 1], by choosing therein $h(x)=h_k(x)=H(x/k)x\in C^1_c(\mathbb{R}^N;\mathbb{R}^N)$, where $H\in C^1_c(\mathbb{R}^N)$ is such that $H(x)=1$ on $|x|\leqslant 1$ and $H(x)=0$ for $|x|\geqslant 2$. Letting $k\to\infty$ and taking into account conditions (5), (6) and that $V'(|x|)|x|\in L^{N/p}(\mathbb{R}^N)$, we reach

$$\int_{\mathbb{R}^N} j_t\big(u_j,|Du_j|\big)|Du_j|-N\int_{\mathbb{R}^N} j\big(u_j,|Du_j|\big)-\frac{N}{p}\int_{\mathbb{R}^N}V\big(|x|\big)|u_j|^p$$
$$+N\lambda_j\int_{\mathbb{R}^N}G(u_j)-\frac{1}{p}\int_{\mathbb{R}^N}V'\big(|x|\big)|x||u_j|^p=0,\quad\text{for all } j\geqslant 1.$$

In turn, each u_j satisfies the following identity

$$f_\lambda(u_j)=\frac{1}{N}\int_{\mathbb{R}^N} j_t\big(u_j,|Du_j|\big)|Du_j|-\frac{1}{Np}\int_{\mathbb{R}^N}V'\big(|x|\big)|x||u_j|^p,\quad\text{for all } j\geqslant 1.$$

Since $f_\lambda(u_j) = c_{\lambda_j}$ and recalling (20) one has

$$\|Du_j\|^p_{L^p(\mathbb{R}^N)}\big(\alpha p\mathcal{S} - \big\|V'\big(|x|\big)|x|\big\|_{L^{N/p}(\mathbb{R}^N)}\big) \leqslant pN\mathcal{S}c_{\lambda_j}, \quad \text{for all } j \geqslant 1,$$

where $\mathcal{S}$ is the best constant for the Sobolev embedding. The last inequality, jointly with (9) and Proposition 7, yields the existence of $A > 0$ such that

$$\|Du_j\|_{L^p(\mathbb{R}^N)} \leqslant A, \quad \text{for all } j \geqslant 1. \tag{23}$$

Also, since u_j solves (22), by testing it with u_j itself (which is admissible), (7) and (20) give

$$\int_{\mathbb{R}^N} V\big(|x|\big)|u_j|^p - \lambda_j \int_{\mathbb{R}^N} g(u_j)u_j \leqslant 0.$$

So that, conditions (8), (18) and (23) yield, for any fixed $\varepsilon < m$,

$$(m - \lambda_j\varepsilon)\|u_j\|^p_{L^p(\mathbb{R}^N)} \leqslant \lambda_j \frac{C_\varepsilon}{\mathcal{S}^{p^*/p}} A^{p^*}. \tag{24}$$

Since (λ_j) is bounded, by combining (23) and (24) we get that (u_j) is bounded in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. In turn, let us observe that (u_j) is a concrete-$(BPS)_{c_1}$ for the functional f_1. In fact notice that, taking into account that $G(u_j)$ remains bounded in $L^1(\mathbb{R}^N)$ due to inequality (18), that $f_{\lambda_j}(u_j) = c_{\lambda_j}$ and recalling Proposition 7, it follows as $j \to \infty$

$$f_1(u_j) = f_{\lambda_j}(u_j) + (\lambda_j - 1)\int_{\mathbb{R}^N} G(u_j) = c_{\lambda_j} + (\lambda_j - 1)\int_{\mathbb{R}^N} G(u_j) = c_1 + o(1). \tag{25}$$

Furthermore, by defining $\hat{w}_j = (\lambda_j - 1)g(u_j) \in W^{-1,p'}(\mathbb{R}^N)$, for every $v \in C_c^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned}\big\langle f_1'(u_j), v\big\rangle &= \int_{\mathbb{R}^N} j_t\big(u_j, |Du_j|\big)\frac{Du_j}{|Du_j|}\cdot Dv + \int_{\mathbb{R}^N} j_s\big(u_j, |Du_j|\big)v + \int_{\mathbb{R}^N} V\big(|x|\big)|u_j|^{p-2}u_j v - \int_{\mathbb{R}^N} g(u_j)v \\ &= \big\langle f_{\lambda_j}'(u_j), v\big\rangle + \langle \hat{w}_j, v\rangle = \langle \hat{w}_j, v\rangle.\end{aligned} \tag{26}$$

Then, since in light of (18) and (23)–(24), $\hat{w}_j \to 0$ in $W^{-1,p'}(\mathbb{R}^N)$ as $j \to \infty$, Proposition 5 applied to f_1 and with $c = c_1$ implies that there exists a function $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that, up to a subsequence, (u_j) converges to u strongly in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. On account of formulas (25)–(26) and the continuity of f_1, and by an application of Lebesgue's Theorem we conclude that u is a nontrivial radial Mountain Pass solution of (10). Finally, u is automatically nonnegative, as follows by testing (10) with the admissible (by [20, Proposition 3.1] holding also for unbounded domains) test function $-u^-$, in view of (7), (20) and the fact that $g(s) = 0$ for every $s \leqslant 0$.

3. Proof of Theorem 2

Eq. (10) is investigated by studying the continuous functional $f_\lambda : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ with $f_\lambda(u)$ again defined as in (13) which, for $\lambda = 1$, corresponds to the action functional associated to (10).

Definition 2. Let $\lambda \in [\delta_0, 1]$, for some $\delta_0 > 0$, and $c \in \mathbb{R}$. We say that f_λ satisfies the concrete-$(SBPS)_c$ condition if every bounded sequence (u_h) in $W^{1,p}(\mathbb{R}^N)$ such that there exists $w_h \in W^{-1,p'}(\mathbb{R}^N)$ with $w_h \to 0$ as $h \to \infty$,

$$f_\lambda(u_h) \to c, \qquad \big\langle f_\lambda'(u_h), v\big\rangle = \langle w_h, v\rangle \quad \forall v \in C_c^\infty\big(\mathbb{R}^N\big),$$

and

$$\big\|u_h - u_h^*\big\|_{L^p(\mathbb{R}^N)\cap L^{p^*}(\mathbb{R}^N)} \to 0, \tag{27}$$

admits a strongly convergent subsequence. Here $u^* := |u|^*$, where $*$ denoted the Schwarz symmetrization.

Proposition 8. *Let $\lambda \in [\delta_0, 1]$, for some $\delta_0 > 0$, $c \in \mathbb{R}$ and assume that* (5)–(8) *and* (14) *hold. Then the functional f_λ satisfies the concrete-$(SBPS)_c$.*

Proof. Given a concrete-$(SBPS)_c$ sequence $(u_h) \subset W^{1,p}(\mathbb{R}^N)$, as in the proof of Proposition 5, up to a subsequence, (u_h) converges to a u weakly, almost everywhere and, in addition, Du_h converges to Du almost everywhere. The main difference with respect to Proposition 5 is that the crucial limit

$$\lim_h \int_{\mathbb{R}^N} g(u_h)u_h = \int_{\mathbb{R}^N} g(u)u, \tag{28}$$

admits now a different justification. Since $(u_h^*) \subset W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ and (u_h) is bounded in $W^{1,p}(\mathbb{R}^N)$, then (u_h^*) is bounded in $W^{1,p}(\mathbb{R}^N)$ too by virtue of the Polya–Szegö inequality. Therefore, since for every $p < q < p^*$ the injection map $i : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ is completely continuous, up to a subsequence, it follows that $u_h^* \to z$ in $L^q(\mathbb{R}^N)$ as $h \to \infty$ for some $z \in L^q(\mathbb{R}^N)$, for $p < q < p^*$. Due to $\|u_h - u_h^*\|_{L^p \cap L^{p^*}(\mathbb{R}^N)} \to 0$ we get $u_h \to z$ in $L^q(\mathbb{R}^N)$, as

$$\|u_h - z\|_{L^q(\mathbb{R}^N)} \leqslant C\|u_h - u_h^*\|_{L^p \cap L^{p^*}(\mathbb{R}^N)} + \|u_h^* - z\|_{L^q(\mathbb{R}^N)}.$$

Of course $z = u$, allowing to conclude that

$$u_h \to u \quad \text{in } L^q(\mathbb{R}^N) \text{ as } h \to \infty, \text{ for every } p < q < p^*. \tag{29}$$

In light of (29), for a $p < q < p^*$ there exists $\zeta \in L^q(\mathbb{R}^N)$, $\zeta \geqslant 0$, such that $|u_h| \leqslant \zeta$ for every $h \geqslant 1$. In turn, by assumption (14), for all $\varepsilon > 0$ there exists $C_\varepsilon \in \mathbb{R}$ with

$$\varepsilon|u_h|^p + C_\varepsilon \zeta^q - g(u_h)u_h \geqslant 0.$$

Then, by Fatou's Lemma, by the arbitrariness of ε and the boundedness of (u_h) in $L^p(\mathbb{R}^N)$,

$$\limsup_h \int_{\mathbb{R}^N} g(u_h)u_h \leqslant \int_{\mathbb{R}^N} g(u)u.$$

Of course, since $g(u_h)u_h \geqslant 0$, again by Fatou's Lemma one also has

$$\liminf_h \int_{\mathbb{R}^N} g(u_h)u_h \geqslant \int_{\mathbb{R}^N} g(u)u,$$

concluding the proof of formula (28). □

Next, we state the main technical tool for the proof of the second theorem.

Lemma 9. *Assume that conditions* (5)–(8) *and* (14)–(15) *hold and that f_λ satisfies the concrete-(SBPS)$_c$ for all $c \in \mathbb{R}$ and all $\lambda \in [\delta_0, 1]$. Then there exists a sequence $(\lambda_j, u_j) \subset [\delta_0, 1] \times W^{1,p}(\mathbb{R}^N)$ with $\lambda_j \nearrow 1$ where u_j is a distributional solution of*

$$-\operatorname{div}\left[j_t\big(u, |Du|\big)\frac{Du}{|Du|}\right] + j_s\big(u, |Du|\big) + V\big(|x|\big)u^{p-1} = \lambda_j g(u) \quad \text{in } \mathbb{R}^N,$$

such that $f_{\lambda_j}(u_j) = c_{\lambda_j}$ and $u_j = u_j^*$.

Proof. The result follows by applying [22, Corollary 3.3] with the following choice of spaces: $X = W^{1,p}(\mathbb{R}^N)$, $S = W^{1,p}(\mathbb{R}^N, \mathbb{R}^+)$ and $V = L^p \cap L^{p^*}(\mathbb{R}^N)$. In fact, it is readily verified that assumptions $(\mathcal{H}_1)$–$(\mathcal{H}_4)$ in [22, Section 3.1] are fulfilled with $u^H = |u|^H$, where v^H denotes the standard polarization of $v \geqslant 0$ and with $u^* = |u|^*$ where v^* denotes the Schwarz symmetrization of $v \geqslant 0$. Condition $(\mathcal{H}_1)$ is just the continuity of the functionals f_λ. Condition $(\mathcal{H}_2)$ is satisfied since Lemma 3 and Lemma 4 hold with the same proof (notice that the function z in the proof of Lemma 3 satisfies $z = z^*$). Condition $(\mathcal{H}_3)$ follows, as in the proof of Lemma 6 by a simple direct computation. Assumption $(\mathcal{H}_4)$ is satisfied by (15) and standard arguments (see also [22, Remark 3.4]). Notice that the function $w = \gamma(1) = z(x/t_0)$ detected in Lemma 3 and used to build the minimax class Γ is radially symmetric and radially decreasing, so that $w^H = w$ for every half space H, as required in $(\mathcal{H}_4)$. □

3.1. *Proof of Theorem 2 concluded*

The proof goes along the lines of the proof of Theorem 1 by simple adaptations of the preparatory results contained in Section 2 to the new setting. With respect to the main differences in the proofs, it is sufficient to replace Proposition 5 with Proposition 8 and Lemma 6 with Lemma 9.

Remark 1. In the notations c and c_{rad} mentioned at the end of the introduction, we always have $c \leqslant c_{\mathrm{rad}}$. On the other hand, when V is constant and the function $t \mapsto j(s,t)$ is p-homogeneous, then $c \geqslant c_{\mathrm{rad}}$. In fact, let u_r be a radial solution at level c provided by Theorem 2, namely $f_1(u_r) = c$. Then, defining the radial curve $\gamma_r(t)(x) := u_r(x/tt_0)$, which belongs to $C([0,1], W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))$ for a suitable $t_0 > 1$ and arguing as in [11, Step I, proof of Theorem 3.2] through the Pohǒzaev identity, it follows that

$$c = f_1(u_r) = \max_{t\in[0,1]} f_1\big(\gamma_r(t)\big),$$

immediately yielding $c \geqslant c_{\mathrm{rad}}$, as desired.

Acknowledgments

The authors wish to thank Jean Van Schaftingen for a useful discussion about the comparison between the Mountain Pass levels c_{rad} and c. They also wish to thank the referee for his/her careful reading of the paper and helpful comments.

References

[1] A. Ambrosetti, P.H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Funct. Anal. 14 (1973) 349–381.
[2] D. Arcoya, L. Boccardo, Critical points for multiple integrals of the calculus of variations, Arch. Ration. Mech. Anal. 134 (1996) 249–274.
[3] A. Azzollini, A. Pomponio, On the Schrödinger equation in under the effect of a general nonlinear term, Indiana Univ. Math. J. 58 (2009) 1361–1378.
[4] H. Berestycki, P.L. Lions, Non-linear scalar field equations. I. Existence of a ground state, Arch. Ration. Mech. Anal. 82 (1983) 313–345.
[5] J. Byeon, L. Jeanjean, M. Maris, Symmetry and monotonicity of least energy solutions, Calc. Var. Partial Differential Equations 36 (2009) 481–492.
[6] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal. 19 (1992) 581–597.
[7] A. Canino, M. Degiovanni, Nonsmooth critical point theory and quasilinear elliptic equations, in: Topological Methods in Differential Equations and Inclusions, Montreal, PQ, 1994, in: NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, Kluwer Acad. Publ., Dordrecht, 1995, pp. 1–50.
[8] J.N. Corvellec, M. Degiovanni, M. Marzocchi, Deformation properties for continuous functionals and critical point theory, Topol. Methods Nonlinear Anal. 1 (1993) 151–171.
[9] M. Degiovanni, M. Marzocchi, A critical point theory for nonsmooth functionals, Ann. Mat. Pura Appl. 167 (1994) 73–100.
[10] M. Degiovanni, A. Musesti, M. Squassina, On the regularity of solutions in the Pucci–Serrin identity, Calc. Var. Partial Differential Equations 18 (2003) 317–334.
[11] A. Giacomini, M. Squassina, Multi-peak solutions for a class of degenerate elliptic equations, Asymptot. Anal. 36 (2003) 115–147.
[12] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999) 787–809.
[13] L. Jeanjean, M. Squassina, Existence and symmetry of least energy solutions for a class of quasi-linear elliptic equations, Ann. Inst. H. Poincaré Anal. Non Linéaire 26 (2009) 1701–1716.
[14] L. Jeanjean, K. Tanaka, A positive solution for a nonlinear Schrödinger equation on $\mathbb{R}^N$, Indiana Univ. Math. J. 54 (2005) 443–464.
[15] L. Jeanjean, J.F. Toland, Bounded Palais–Smale mountain-pass sequences, C. R. Math. Acad. Sci. Paris Sér. I 327 (1998) 23–28.
[16] O. Ladyzenskaya, N. Uraltseva, Linear and Quasilinear Elliptic Equations, Academic Press, 1968.
[17] P.L. Lions, The concentration–compactness principle in the calculus of variations. The locally compact case. Part I and II, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 109–145 and 223–283.
[18] B. Pellacci, Critical points for non-differentiable functionals, Boll. Unione Mat. Ital. B 11 (1997) 733–749.
[19] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.
[20] M. Squassina, Weak solutions to general Euler's equations via non-smooth critical point theory, Ann. Fac. Sci. Toulouse Math. 9 (2000) 113–131.
[21] M. Squassina, On the Palais principle for nonsmooth functionals, Nonlinear Anal., in press.
[22] M. Squassina, On Struwe–Jeanjean–Toland monotonicity trick, Proc. Roy. Soc. Edinburgh Sect. A, in press.
[23] M. Squassina, Existence, multiplicity, perturbation, and concentration results for a class of quasi-linear elliptic problems, in: Electron. J. Differ. Equ. Monogr., vol. 7, Texas State University of San Marcos, TX, USA, 2006, 213 pp.
[24] M. Struwe, Variational Methods. Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, fourth edition, Springer-Verlag, Berlin, 2008, 302 pp.
[25] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations 51 (1984) 126–150.