Perturbed $S^1$-Symmetric Hamiltonian Systems

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Abstract—By techniques of critical point theory, we show that multiple periodic weak solutions of a general class of Hamiltonian systems persist despite perturbation with an $L^2$ term destroying the $S^1$-symmetries. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, in the spirit of [1], we want to investigate the effect of perturbing the $S^1$-symmetry of a general class of Hamiltonian systems.

Studied around 1980 by Bahri and Berestycki in [2], the problem of finding multiple periodic solutions of nonsymmetric systems of type $\left( T_{ij} (s) = \delta_{ij} \right)$

$$\begin{align*}
\dot{\gamma}_t &= D_s \gamma V (\gamma) + \varphi_t, \quad \ell = 1, \ldots, n,
\end{align*}$$

where $\varphi \in L^2(S^1, \mathbb{R}^n)$, have also been considered by Rabinowitz in [3] via techniques of classical critical point theory.

In order to find weak solutions to (1.1), he looked for critical points of the smooth ($C^1$) action $\mathcal{L} : H^1(S^1, \mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) = \frac{1}{2} \int_0^{2\pi} |\gamma|^2 \, d\tau - \int_0^{2\pi} V (\gamma) \, d\tau - \int_0^{2\pi} \varphi \cdot \gamma \, d\tau.$$

On the other hand, the action of a mechanical system with $n$ degree of freedom, in general, may be represented by quasi-linear functionals $\mathcal{L} : H^1(S^1, \mathbb{R}^n) \to \mathbb{R}$ of the type

$$\mathcal{L}(\gamma) = \frac{1}{2} \int_0^{2\pi} \sum_{i,j=1}^n T_{ij} (\gamma) \dot{\gamma}_i \dot{\gamma}_j \, d\tau - \int_0^{2\pi} V (\gamma) \, d\tau - \int_0^{2\pi} \varphi \cdot \gamma \, d\tau,$$

where $T_{ij} (s) = \delta_{ij}$.
where \( \{T_{ij}(s)\} \) is the symmetric positive definite quadratic form of kinetic energy and \( V \) is the potential energy. If \( \varphi = 0 \), clearly for each \( \gamma \in H^1(S^1, \mathbb{R}^n) \) we have

\[
\forall \vartheta \in \mathbb{R}, \quad L(T_\gamma \vartheta) = L(\gamma), \quad (T_\gamma \vartheta)(\tau) = \gamma(\tau + \vartheta), \quad (S^1\text{-symmetry}).
\]

If \( \varphi \neq 0 \), the \( S^1 \)-symmetry drops and the associated evolution system is given by

\[
- \sum_{i=1}^{n} (T^{ij}(\gamma) \dot{\gamma}_i)' + \frac{1}{2} \sum_{i,j=1}^{n} D_{s_{ii}} T^{ij}(\gamma) \dot{\gamma}_i \dot{\gamma}_j = D_{s_{ii}} V(\gamma) + \varphi, \quad (1.3)
\]

for \( \ell = 1, \ldots, n \). Now, since \( L^1(S^1, \mathbb{R}^n) \subseteq H^{-1}(S^1, \mathbb{R}^n) \), (1.2) is a smooth functional and we shall apply the techniques of classical critical point theory [3-5].

Recently, some papers have been published about the existence of weak solutions to quasi-linear elliptic systems subjected to perturbation from \( Z_2 \)-symmetry, \( L(-\gamma) = L(\gamma) \). See [6,7]. On the other hand, to my knowledge, little is known for \( S^1 \)-symmetries in case of the quasi-linear functional (1.2).

Throughout the paper, we shall consider the following assumptions.

(i) \( T^{ij}(\cdot) \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) and \( D_{s_{ii}} T^{ij}(\cdot) \in L^\infty(\mathbb{R}^n) \) for each \( i, j = 1, \ldots, n \). Moreover,

\[
\sum_{i,j=1}^{n} T^{ij}(s) \xi_i \xi_j \geq \nu |\xi|^2, \quad (\nu > 0), \quad (1.4)
\]

for each \( (s, \xi) \in \mathbb{R}^{2n} \).

(ii) \( V \in C^1(\mathbb{R}^n) \) and there exist \( b_1, b_2, R > 0 \) and \( \sigma, \mu > 2 \) such that

\[
V(s) \leq b_1 + b_2 |s|^\mu, \quad |s| \geq R \implies 0 < \mu V(s) \leq s \cdot \nabla V(s), \quad (1.5)
\]

(1.6)

for each \( s \in \mathbb{R}^n \). Finally, there exists \( \theta \in [0, \mu - 2] \) such that

\[
\sum_{i,j=1}^{n} s \cdot D_{s} T^{ij}(s) \xi_i \xi_j \leq \theta \sum_{i,j=1}^{n} T^{ij}(s) \xi_i \xi_j, \quad (1.7)
\]

for each \( (s, \xi) \in \mathbb{R}^{2n} \).

Under the previous assumptions, the following is our main result.

**THEOREM 1.** Let \( \varphi \in L^2(S^1, \mathbb{R}^n) \) and assume that

\[
\sigma < 4\mu - 2.
\]

Then the perturbed Hamiltonian system \( (\ell = 1, \ldots, n) \)

\[
- \sum_{i=1}^{n} (T^{ij}(\gamma) \dot{\gamma}_i)' + \frac{1}{2} \sum_{i,j=1}^{n} D_{s_{ii}} T^{ij}(\gamma) \dot{\gamma}_i \dot{\gamma}_j = D_{s_{ii}} V(\gamma) + \varphi \quad (1.7)
\]

admits a sequence \( (\gamma_h) \) of weak solutions in \( H^1(S^1, \mathbb{R}^n) \).

This result extends Theorem 2.4 in [3] to a more general class of Hamiltonian systems.
2. PERTURBED $S^1$-SYMMETRIC FUNCTIONALS

By condition (1.6), we find $c_1, c_2, c_3 > 0$ such that for each $s \in \mathbb{R}^n$,

$$\frac{1}{\mu} (s \cdot \nabla V(s) + c_1) \geq V(s) + c_2 \geq c_3 |s|^\mu. \quad (2.8)$$

**Lemma 1.** If $\gamma \in H^1(S^1, \mathbb{R}^n)$ is a weak solution to (1.7), there exists $c > 0$ with

$$\int_0^{2\pi} (V(\gamma) + c_2) \, d\tau \leq c \left( \mathcal{L}(\gamma)^2 + 1 \right)^{1/2}. \quad (2.9)$$

**Proof.** It suffices to follow the steps of Lemma 2.1 in [1].

Let us now define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi(\tau) = 1$ for $\tau \leq 1$, $\chi(\tau) = 0$ for $\tau \geq 2$, and $-2 < \chi'(\tau) < 0$ when $1 < \tau < 2$, and let for each $\gamma \in H^1(S^1, \mathbb{R}^n)$

$$\phi(\gamma) = 2c \left( \mathcal{L}(\gamma)^2 + 1 \right)^{1/2}, \quad \psi(\gamma) = \chi \left( \phi(\gamma)^{-1} \int_0^{2\pi} (V(\gamma) + c_2) \, d\tau \right).$$

Finally, we define the modified functional by

$$\tilde{\mathcal{L}}(\gamma) = \frac{1}{2} \int_0^{2\pi} \sum_{i,j=1}^n T^{ij}(\gamma) \hat{\gamma}_i \hat{\gamma}_j \, d\tau - \int_0^{2\pi} V(\gamma) \, d\tau - \psi(\gamma) \int_0^{2\pi} \varphi \cdot \gamma \, d\tau. \quad (2.10)$$

The Euler's equation associated with the previous functional is given by

$$-\sum_{i=1}^n \left( T^{il}(\gamma) \hat{\gamma}_l \right)' + \frac{1}{2} \sum_{i,j=1}^n D_{sr} T^{ij}(\gamma) \hat{\gamma}_i \hat{\gamma}_j = D_{sr} \tilde{V}(\gamma), \quad (2.10)$$

where

$$\nabla \tilde{V}(\gamma) = \nabla V(\gamma) + \psi(\gamma) \varphi + \psi'(\gamma) \int_0^{2\pi} \varphi \cdot \gamma \, d\tau.$$

Note that, by the previous lemma, if $\gamma \in H^1(S^1, \mathbb{R}^n)$ is a weak solution to (1.7), we have that $\psi(\gamma) = 1$, and therefore, $\tilde{\mathcal{L}}(\gamma) = \mathcal{L}(\gamma)$.

The next result measures the defect of $S^1$-symmetry of $\tilde{\mathcal{L}}$. This turns out to be crucial in the final comparison argument.

**Lemma 2.** There exists $\beta > 0$ such that for all $\gamma \in H^1(S^1, \mathbb{R}^n)$

$$|\tilde{\mathcal{L}}(\gamma) - \tilde{\mathcal{L}}(T_\theta \gamma)| \leq \beta \left( \left| \tilde{\mathcal{L}}(\gamma) \right|^{1/\mu} + 1 \right).$$

**Proof.** Taking into account Lemma 1 and the fact that $\|\gamma\|_2 = \|T_\theta(\gamma)\|_2$, the proof follows as in [1, Lemma 2.6].

**Theorem 2.** There exists $M > 0$ such that if $\gamma \in H^1(S^1, \mathbb{R}^n)$ is a weak solution to (2.10) with $\tilde{\mathcal{L}}(\gamma) \geq M$, then $\gamma$ is a weak solution to (1.7) and $\tilde{\mathcal{L}}(\gamma) = \mathcal{L}(\gamma)$.

**Proof.** Follow the steps of Theorem 2.3 in [1].
3. THE PALAIS-SMALE CONDITION FOR $\tilde{\mathcal{L}}$

**Definition 1.** If $c \in \mathbb{R}$, a sequence $(\gamma^h) \subseteq H^1(S^1, \mathbb{R}^n)$ is said to be a Palais-Smale sequence at level $c$ ((PS)$_c$-sequence, in short) for $\tilde{\mathcal{L}}$, if $\tilde{\mathcal{L}}(\gamma^h) \to c$, and (for $\ell = 1, \ldots, n$)

$$-\sum_{i=1}^n (T^{i\ell}(\gamma)\gamma^h_i) + \frac{1}{2} \sum_{i,j=1}^n D_s T^{ij}(\gamma)v_i \nabla_j - D_s \nabla v(\gamma) \to 0,$$

strongly in $H^{-1}(S^1)$.

We say that $\tilde{\mathcal{L}}$ satisfies the Palais-Smale condition at level $c$, if each (PS)$_c$ sequence for $\tilde{\mathcal{L}}$ has a strongly convergent subsequence in $H^1(S^1, \mathbb{R}^n)$.

**Lemma 3.** There exists $M > 0$ such that each (PS)$_c$-sequence $(\gamma^h)$ for $\tilde{\mathcal{L}}$ with $c \geq M$ is bounded in $H^1(S^1, \mathbb{R}^n)$.

**Proof.** Let $M > 0$ and $(\gamma^h)$ be a (PS)$_c$-sequence for $\tilde{\mathcal{L}}$ with $c \geq M$ such that $M \leq \tilde{\mathcal{L}}(\gamma^h) \leq K$, for some $K > 0$ and $h \in \mathbb{N}$ large. By Lemma 3 in [7], we have

$$\lim_{h} \tilde{\mathcal{L}}'(\gamma^h)(\gamma^h) = 0.$$

Therefore, arguing as in [1, Lemma 3.2], for large $h \in \mathbb{N}$ and any $\varphi > 0$, it follows

$$\varphi \left\| \gamma^h \right\|_{1,2} + K \geq \tilde{\mathcal{L}}(\gamma^h) - \varphi \tilde{\mathcal{L}}'(\gamma^h)(\gamma^h) \geq \frac{\nu}{2} \left(1 - \varphi(2 + \theta)(1 + T_1(\gamma^h))\right) \left\| \gamma^h \right\|_{1,2}^2$$

$$+ (\mu\varphi(1 + T_2(\gamma^h)) - 1) \int_0^{2\pi} V(\gamma^h) d\tau - \varphi \left(1 + T_1(\gamma^h)\right) + 1 \left\| \varphi \right\|_{2} \left\| \gamma^h \right\|_{2},$$

where $\nu > 0$ is the ellipticity constant of coefficients $T^{ij}$ and $T_1, T_2 : H^1(S^1, \mathbb{R}^n) \to \mathbb{R}$ are defined by setting

$$T_1(\gamma) = \chi'(\vartheta(\gamma)) (2c)^2 \vartheta(\gamma)^2 \int_0^{2\pi} \varphi \cdot \gamma d\tau,$$

$$T_2(\gamma) = \chi'(\vartheta(\gamma)) \vartheta(\gamma)^{-1} \int_0^{2\pi} \varphi \cdot \gamma d\tau + T_1(\gamma), \quad \vartheta(\gamma) := \vartheta(\gamma)^{-1} \int_0^{2\pi} (V(\gamma) + c_2) d\tau.$$

If we choose $M$ sufficiently large, we find $\varepsilon > 0$, $\eta > 0$, and $\varphi \in \{1 + \eta, \mu(1 - \varepsilon)/(\theta + 2)\}$ such that uniformly in $h \in \mathbb{N}$

$$(1 - \varphi(2 + \theta)(1 + T_1(\gamma^h))) \geq \varepsilon, \quad (\mu\varphi(1 + T_2(\gamma^h)) - 1) \geq \eta.$$

Hence, we obtain, for some $b > 0$ and $c > 0$,

$$\varphi \left\| \gamma^h \right\|_{1,2} + K \geq \frac{\nu\varepsilon}{2} \left\| \gamma^h \right\|_{1,2}^2 + b\eta \left\| \gamma^h \right\|_\mu^2 - c \left\| \gamma^h \right\|_{1,2},$$

which implies the boundedness of $(\gamma^h)$ in $H^1(S^1, \mathbb{R}^n)$. 

We now recall a crucial property for the Palais-Smale condition to hold.

**Lemma 4.** Let $(\gamma^h)$ be a bounded sequence in $H^1(S^1, \mathbb{R}^n)$ and set

$$\langle w^h, \eta \rangle = \int_0^{2\pi} \sum_{i,j=1}^n T^{ij}(\gamma^h) \chi_i^h \eta_j^h d\tau + \frac{1}{2} \int_0^{2\pi} \sum_{i,j=1}^n D_s T^{ij}(\gamma^h) \cdot \eta_i^h \eta_j^h d\tau,$$

for all $\eta \in C_0^{\infty}(S^1, \mathbb{R}^n)$. Then, if $(w^h)$ is strongly convergent to some $w$ in $H^{-1}(S^1, \mathbb{R}^n)$, $(\gamma^h)$ admits a strongly convergent subsequence in $H^1(S^1, \mathbb{R}^n)$.

**Proof.** Since in our setting $L^1(S^1, \mathbb{R}^n) \subseteq H^{-1}(S^1, \mathbb{R}^n)$, the proof is standard.

We point out that the previous lemma is absolutely nontrivial in more than one variable. See [1,7].

We now come to one of the main tools of this paper.

**Theorem 3.** There is $M > 0$ such that $\tilde{\mathcal{L}}$ satisfies the (PS)$_c$-condition for $c \geq M$.

**Proof.** Taking into account Lemma 3.3 of [1], combine Lemma 3 and Lemma 4.
4. EXISTENCE OF MULTIPLE PERIODIC ORBITS

Now, let \( \{e_1, \ldots, e_n\} \) be the standard basis in \( \mathbb{R}^n \) and define, for each \( 1 \leq i \leq n \),
\[
E_{m,i} := \text{span} \{ v_{j,k} = \sin (j\tau) e_k, v_{j,k} = \cos (j\tau) e_k : 1 \leq j \leq m, 1 \leq k \leq i \}.
\]

By inequality (2.8), there exists \( R_{m,i} > 0 \) such that
\[
\forall \gamma \in E_{m,i}, \quad \| \gamma \|_{1,2} \geq R_{m,i} \implies \tilde{L}(\gamma) \leq 0.
\]

If \( D_{m,i} = B_{R_{m,i}} \cap E_{m,i} \), we say that \( \eta \in C(D_{m,i}, H^1(S^1, \mathbb{R}^n)) \) is equivariant if
\[
\forall \theta \in [0, 2\pi[, \quad \eta(\theta(\gamma)) = \theta \eta(\gamma).
\]

Finally, set
\[
\Gamma_{k,i} = \{ \eta \in C(D_{k,i}, H^1) : \eta \text{ equiv. } \eta(\gamma) = \gamma \text{ if } \| \gamma \| = R_{k,i} \text{ or } \gamma \in E_{0,n} \}
\]
and
\[
b_{k,i} = \inf_{\eta \in \Gamma_{k,i}} \max_{\eta \in D_{k,i}} \tilde{L}(\eta(\gamma)).
\]

The following result depends on an \( S^1 \)-version of Borsuk-Ulam's Theorem.

**Lemma 5.** For each \( k \in \mathbb{N}, 1 \leq i \leq n, \varrho \in [0, R_{k,i}], \) and \( \eta \in \Gamma_{k,i} \),
\[
\eta(D_{k,i}) \cap \partial B(0, \varrho) \cap E_{k,i-1}^1 \neq \emptyset.
\]

**Proof.** See [3, Lemma 2.20].

**Lemma 6.** There exist \( \beta > 0 \) and \( k_0 \in \mathbb{N} \) such that, for each \( 1 \leq i \leq n, \)
\[
\forall k \geq k_0, \quad b_{k,i} \geq \beta k^{(\sigma+2)/(\sigma-2)}.
\]

**Proof.** If \( k \geq 1 \) and \( \gamma \in \partial B(0, \varrho) \cap E_{k,i-1}^1 \), arguing as in [1, Lemma 5.3], by (1.5), we have
\[
\tilde{L}(\gamma) \geq \nu \frac{\varrho^2}{4} - \alpha_1 \| \gamma \|_{\sigma}^\sigma - \alpha_2 - \alpha_3 \| \gamma \|_2^2.
\]

for some \( \alpha_1, \alpha_2, \alpha_3 > 0 \). Now, Gagliardo-Niremberg's inequality implies that
\[
\| \gamma \|_\sigma \leq \alpha_4 \| \gamma \|_1^{(\sigma-2)/2\sigma} \| \gamma \|_2^{(\sigma+2)/2\sigma},
\]
for some \( \alpha_4 > 0 \) and all \( \gamma \in H^1 \). Moreover,
\[
\| \gamma \|_2 \leq \frac{1}{k} \| \gamma \|_2,
\]
for each \( \gamma \in E_{k,i-1}^1 \). Continuing as in [1, Lemma 5.3], we conclude the proof.

For each \( k \in \mathbb{N} \), we now set
\[
U_{k,i} = \{ \gamma = \tau v_{k,i+1} + \phi : \tau \in [0, R_{k,i+1}], \phi \in B(0, R_{k,i+1}) \cap E_{k,i}, \| \gamma \|_{1,2} \leq R_{k,i+1} \},
\]
\[
\Lambda_{k,i} = \{ \lambda \in C(U_{k,i}, H^1) : \lambda|_{D_{k,i}} \in \Gamma_{k,i}, \lambda|_{\partial B(0, R_{k,i}) \cup B(0, R_{k,i}) \cap E_{k,i}} = Id \}.
\]

We now recall the main existence tool from critical point theory.
Theorem 4. Assume that $c_{k,i} > b_{k,i} \geq M$. If $0 < \delta < c_{k,i} - b_{k,i}$ and

$$\Lambda_{k,i}(\delta) := \left\{ \lambda \in \Lambda_{k,i} : \tilde{L}(\lambda(\gamma)) \leq b_{k,i} + \delta, \text{ for } \gamma \in D_{k,i} \right\},$$

set

$$c_{k,i}(\delta) = \inf_{\lambda \in \Lambda_{k,i}(\delta)} \max_{\gamma \in U_{k,i}} \tilde{L}(\lambda(\gamma)).$$

Then $c_{k,i}(\delta)$ is a critical value for $\tilde{L}$.

Proof. Argue essentially as in [3, Lemma 2.29].

It only remains to show that condition $c_{k,i} = b_{k,i}$ for $k$ large is not permitted.

Lemma 7. Assume that $c_{k,i} = b_{k,i}$ for all $k \geq k_1$ and $1 \leq i \leq n$. Then, there exist $\gamma > 0$ and $k \geq k_1$ with

$$b_{k,i} \leq \gamma\lambda_{k,i}/(\mu-1).$$

Proof. Choose $k \geq k_1$, $1 \leq i \leq n$, and $\epsilon > 0$ and let $\lambda \in \Lambda_{k,i}$ be such that

$$\max_{\gamma \in U_{k,i}} \tilde{L}(\lambda(\gamma)) \leq b_{k,i} + \epsilon.$$

Now, let $\tilde{\lambda}(\gamma) = \lambda(\gamma)$ and $\tilde{\lambda}(\gamma) = T_{\gamma}\lambda(\gamma)$ for $\gamma \in U_{k,i}$ It is easy to show that $\tilde{\lambda} \in \Gamma_{k,i+1}$. Then, arguing as in [1, Lemma 5.6], we get

$$b_{k,i+1} \leq b_{k,i} + \beta \left( \left| b_{k,i} \right|^{1/\mu} + 1 \right),$$

for $k \geq k_1$. The proof now goes on as in [3, Lemma 2.31].

Finally, we come to the proof of Theorem 1. Since $\sigma < 4\mu - 2$ implies that $\mu/\mu - 1 < (\sigma + 2)/(\sigma - 2)$, combining Lemmas 6 and 7, we deduce, by Theorem 4 that $(c_{k,i}(\delta))$ is a sequence of critical values for $\tilde{L}$. Whence, by Theorem 4, $\tilde{L}$ has a sequence of critical values.

References