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Perturbed $\mathbb{S}^1$-Symmetric Hamiltonian Systems

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Abstract—By techniques of critical point theory, we show that multiple periodic weak solutions of a general class of Hamiltonian systems persist despite perturbation with an L^2 term destroying the $\mathbb{S}^1$-symmetries.

Keywords—Palais-Smale condition, Critical point theory, Hamiltonian systems, Perturbation from symmetry, Multiple periodic solutions.

1. INTRODUCTION

In this paper, in the spirit of [1], we want to investigate the effect of perturbing the $\mathbb{S}^1$-symmetry of a general class of Hamiltonian systems.

Studied around 1980 by Bahri and Berestycki in [2], the problem of finding multiple periodic solutions of nonsymmetric systems of type ($T^{ij}(s) = \delta_{ij}$)

$$\ddot{\gamma}_\ell = D_{s_\ell} V(\gamma) + \varphi_\ell, \qquad \ell = 1, \dots, n, \tag{1.1}$$

where $\varphi \in L^2(\mathbb{S}^1, \mathbb{R}^n)$, have also been considered by Rabinowitz in [3] via techniques of classical critical point theory.

In order to find weak solutions to (1.1), he looked for critical points of the smooth (C^1) action $\mathcal{L} : H^1(\mathbb{S}^1, \mathbb{R}^n) \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) = \frac{1}{2} \int_0^{2\pi} |\dot{\gamma}|^2 \, d\tau - \int_0^{2\pi} V(\gamma) \, d\tau - \int_0^{2\pi} \varphi \cdot \gamma \, d\tau.$$

On the other hand, the action of a mechanical system with n degree of freedom, in general, may be represented by quasi-linear functionals $\mathcal{L} : H^1(\mathbb{S}^1, \mathbb{R}^n) \to \mathbb{R}$ of the type

$$\mathcal{L}(\gamma) = \frac{1}{2} \int_0^{2\pi} \sum_{i,j=1}^n T^{ij}(\gamma) \dot{\gamma}_i \dot{\gamma}_j \, d\tau - \int_0^{2\pi} V(\gamma) \, d\tau - \int_0^{2\pi} \varphi \cdot \gamma \, d\tau, \tag{1.2}$$

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where $\{T^{ij}(s)\}$ is the symmetric positive definite quadratic form of kinetic energy and V is the potential energy. If $\varphi = 0$, clearly for each $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^n)$ we have

$$\forall\, \vartheta \in \mathbb{R}, \qquad \mathcal{L}(T_\vartheta \gamma) = \mathcal{L}(\gamma), \quad (T_\vartheta \gamma)(\tau) = \gamma(\tau + \vartheta), \qquad (\mathbb{S}^1\text{-symmetry}).$$

If $\varphi \neq 0$, the $\mathbb{S}^1$-symmetry drops and the associated evolution system is given by

$$-\sum_{i=1}^{n} \left(T^{i\ell}(\gamma)\dot\gamma_i\right)' + \frac{1}{2}\sum_{i,j=1}^{n} D_{s_\ell} T^{ij}(\gamma)\dot\gamma_i\dot\gamma_j = D_{s_\ell} V(\gamma) + \varphi_\ell, \tag{1.3}$$

for $\ell = 1, \dots, n$. Now, since $L^1(\mathbb{S}^1, \mathbb{R}^n) \subseteq H^{-1}(\mathbb{S}^1, \mathbb{R}^n)$, (1.2) is a smooth functional and we shall apply the techniques of classical critical point theory [3–5].

Recently, some papers have been published about the existence of weak solutions to quasi-linear elliptic systems subjected to perturbation from $\mathbb{Z}_2$-symmetry $(\mathcal{L}(-\gamma) = \mathcal{L}(\gamma))$. See [6,7]. On the other hand, to my knowledge, little is known for $\mathbb{S}^1$-symmetries in case of the quasi-linear functional (1.2).

Throughout the paper, we shall consider the following assumptions.

(i) $T^{ij}(\cdot) \in C^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and $D_s T^{ij}(\cdot) \in L^\infty(\mathbb{R}^n)$ for each $i, j = 1, \dots, n$. Moreover,

$$\sum_{i,j=1}^{n} T^{ij}(s)\xi_i\xi_j \geq \nu\,|\xi|^2, \qquad (\nu > 0), \tag{1.4}$$

for each $(s, \xi) \in \mathbb{R}^{2n}$.

(ii) $V \in C^1(\mathbb{R}^n)$ and there exist $b_1, b_2, R > 0$ and $\sigma, \mu > 2$ such that

$$V(s) \leq b_1 + b_2|s|^\sigma, \tag{1.5}$$

$$|s| \geq R \Longrightarrow 0 < \mu V(s) \leq s \cdot \nabla V(s), \tag{1.6}$$

for each $s \in \mathbb{R}^n$. Finally, there exists $\theta \in]0, \mu - 2[$ such that

$$\sum_{i,j=1}^{n} s \cdot D_s T^{ij}(s)\xi_i\xi_j \leq \theta \sum_{i,j=1}^{n} T^{ij}(s)\xi_i\xi_j,$$

for each $(s, \xi) \in \mathbb{R}^{2n}$.

Under the previous assumptions, the following is our main result.

THEOREM 1. *Let $\varphi \in L^2(\mathbb{S}^1, \mathbb{R}^n)$ and assume that*

$$\sigma < 4\mu - 2.$$

Then the perturbed Hamiltonian system ($\ell = 1, \dots, n$)

$$-\sum_{i=1}^{n} \left(T^{i\ell}(\gamma)\dot\gamma_i\right)' + \frac{1}{2}\sum_{i,j=1}^{n} D_{s_\ell} T^{ij}(\gamma)\dot\gamma_i\dot\gamma_j = D_{s_\ell} V(\gamma) + \varphi_\ell \tag{1.7}$$

admits a sequence (γ_h) of weak solutions in $H^1(\mathbb{S}^1, \mathbb{R}^n)$.

This result extends Theorem 2.4 in [3] to a more general class of Hamiltonian systems.

2. PERTURBED $\mathbb{S}^1$-SYMMETRIC FUNCTIONALS

By condition (1.6), we find $c_1, c_2, c_3 > 0$ such that for each $s \in \mathbb{R}^n$,

$$\frac{1}{\mu}(s \cdot \nabla V(s) + c_1) \geq V(s) + c_2 \geq c_3 |s|^\mu. \tag{2.8}$$

LEMMA 1. *If $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^n)$ is a weak solution to (1.7), there exists $c > 0$ with*

$$\int_0^{2\pi} (V(\gamma) + c_2)\, d\tau \leq c \left(\mathcal{L}(\gamma)^2 + 1\right)^{1/2}.$$

PROOF. It suffices to follow the steps of Lemma 2.1 in [1]. ∎

Let us now define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi(\tau) = 1$ for $\tau \leq 1$, $\chi(\tau) = 0$ for $\tau \geq 2$, and $-2 < \chi'(\tau) < 0$ when $1 < \tau < 2$, and let for each $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^n)$

$$\phi(\gamma) = 2c\left(\mathcal{L}(\gamma)^2 + 1\right)^{1/2}, \qquad \psi(\gamma) = \chi\left(\phi(\gamma)^{-1} \int_0^{2\pi} (V(\gamma) + c_2)\, d\tau\right).$$

Finally, we define the modified functional by

$$\tilde{\mathcal{L}}(\gamma) = \frac{1}{2}\int_0^{2\pi} \sum_{i,j=1}^n T^{ij}(\gamma)\dot{\gamma}_i\dot{\gamma}_j\, d\tau - \int_0^{2\pi} V(\gamma)\, d\tau - \psi(\gamma)\int_0^{2\pi} \varphi \cdot \gamma\, d\tau. \tag{2.9}$$

The Euler's equation associated with the previous functional is given by

$$-\sum_{i=1}^n \left(T^{i\ell}(\gamma)\dot{\gamma}_i\right)' + \frac{1}{2}\sum_{i,j=1}^n D_{s_\ell}T^{ij}(\gamma)\dot{\gamma}_i\dot{\gamma}_j = D_{s_\ell}\tilde{V}(\gamma), \tag{2.10}$$

where

$$\nabla\tilde{V}(\gamma) = \nabla V(\gamma) + \psi(\gamma)\varphi + \psi'(\gamma)\int_0^{2\pi} \varphi \cdot \gamma\, d\tau.$$

Note that, by the previous lemma, if $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^n)$ is a weak solution to (1.7), we have that $\psi(\gamma) = 1$, and therefore, $\tilde{\mathcal{L}}(\gamma) = \mathcal{L}(\gamma)$.

The next result measures the defect of $\mathbb{S}^1$-symmetry of $\tilde{\mathcal{L}}$. This turns out to be crucial in the final comparison argument.

LEMMA 2. *There exists $\beta > 0$ such that for all $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^n)$*

$$\left|\tilde{\mathcal{L}}(\gamma) - \tilde{\mathcal{L}}(T_\vartheta\gamma)\right| \leq \beta\left(\left|\tilde{\mathcal{L}}(\gamma)\right|^{1/\mu} + 1\right).$$

PROOF. Taking into account Lemma 1 and the fact that $\|\gamma\|_2 = \|T_\vartheta(\gamma)\|_2$, the proof follows as in [1, Lemma 2.6]. ∎

THEOREM 2. *There exists $M > 0$ such that if $\gamma \in H^1(\mathbb{S}^1, \mathbb{R}^n)$ is a weak solution to (2.10) with $\tilde{\mathcal{L}}(\gamma) \geq M$, then γ is a weak solution to (1.7) and $\tilde{\mathcal{L}}(\gamma) = \mathcal{L}(\gamma)$.*

PROOF. Follow the steps of Theorem 2.3 in [1]. ∎

3. THE PALAIS-SMALE CONDITION FOR $\tilde{\mathcal{L}}$

DEFINITION 1. *If $c \in \mathbb{R}$, a sequence $(\gamma^h) \subseteq H^1(\mathbb{S}^1, \mathbb{R}^n)$ is said to be a Palais-Smale sequence at level c ($(\mathrm{PS})_c$-sequence, in short) for $\tilde{\mathcal{L}}$, if $\tilde{\mathcal{L}}(\gamma^h) \to c$, and (for $\ell = 1, \dots, n$)*

$$-\sum_{i=1}^{n} \left(T^{i\ell}(\gamma)\dot{\gamma}_i\right)' + \frac{1}{2}\sum_{i,j=1}^{n} D_{s_\ell} T^{ij}(\gamma)\dot{\gamma}_i\dot{\gamma}_j - D_{s_\ell}\tilde{V}(\gamma) \to 0,$$

strongly in $H^{-1}(\mathbb{S}^1)$.

We say that $\tilde{\mathcal{L}}$ satisfies the Palais-Smale condition at level c, if each $(\mathrm{PS})_c$ sequence for $\tilde{\mathcal{L}}$ has a strongly convergent subsequence in $H^1(\mathbb{S}^1, \mathbb{R}^n)$.

LEMMA 3. *There exists $M > 0$ such that each $(\mathrm{PS})_c$-sequence (γ^h) for $\tilde{\mathcal{L}}$ with $c \geq M$ is bounded in $H^1(\mathbb{S}^1, \mathbb{R}^n)$.*

PROOF. Let $M > 0$ and (γ^h) be a $(\mathrm{PS})_c$-sequence for $\tilde{\mathcal{L}}$ with $c \geq M$ such that $M \leq \tilde{\mathcal{L}}(\gamma^h) \leq K$, for some $K > 0$ and $h \in \mathbb{N}$ large. By Lemma 3 in [7], we have

$$\lim_h \tilde{\mathcal{L}}'\left(\gamma^h\right)\left(\gamma^h\right) = 0.$$

Therefore, arguing as in [1, Lemma 3.2], for large $h \in \mathbb{N}$ and any $\varrho > 0$, it follows

$$\begin{aligned}\varrho\left\|\gamma^h\right\|_{1,2} + K \geq \tilde{\mathcal{L}}\left(\gamma^h\right) - \varrho\tilde{\mathcal{L}}'\left(\gamma^h\right)\left(\gamma^h\right) &\geq \frac{\nu}{2}\left(1 - \varrho(2+\theta)\left(1 + T_1\left(\gamma^h\right)\right)\right)\left\|\gamma^h\right\|_{1,2}^2 \\ &+ \left(\mu\varrho\left(1 + T_2\left(\gamma^h\right)\right) - 1\right)\int_0^{2\pi} V\left(\gamma^h\right)\, d\tau - \left[\varrho\left(1 + T_1\left(\gamma^h\right)\right) + 1\right]\|\varphi\|_2\left\|\gamma^h\right\|_2,\end{aligned}$$

where $\nu > 0$ is the ellipticity constant of coefficients T^{ij} and $T_1, T_2 : H^1(\mathbb{S}^1, \mathbb{R}^n) \to \mathbb{R}$ are defined by setting

$$T_1(\gamma) = \chi'(\vartheta(\gamma))(2c)^2\vartheta(u)\phi(u)^{-2}\mathcal{L}(\gamma)\int_0^{2\pi}\varphi\cdot\gamma\, d\tau,$$

$$T_2(\gamma) = \chi'(\vartheta(\gamma))\phi(\gamma)^{-1}\int_0^{2\pi}\varphi\cdot\gamma\, d\tau + T_1(\gamma), \qquad \vartheta(\gamma) := \phi(\gamma)^{-1}\int_0^{2\pi}(V(\gamma) + c_2)\, d\tau.$$

If we choose M sufficiently large, we find $\varepsilon > 0$, $\eta > 0$, and $\varrho \in](1+\eta)/\mu, (1-\varepsilon)/(\theta+2)[$ such that uniformly in $h \in \mathbb{N}$

$$\left(1 - \varrho(2+\theta)\left(1 + T_1\left(\gamma^h\right)\right)\right) > \varepsilon, \qquad \left(\mu\varrho\left(1 + T_2\left(\gamma^h\right)\right) - 1\right) > \eta.$$

Hence, we obtain, for some $b > 0$ and $c > 0$,

$$\varrho\left\|\gamma^h\right\|_{1,2} + K \geq \frac{\nu\varepsilon}{2}\left\|\gamma^h\right\|_{1,2}^2 + b\eta\left\|\gamma^h\right\|_\mu^\mu - c\left\|\gamma^h\right\|_{1,2},$$

which implies the boundedness of (γ^h) in $H^1(\mathbb{S}^1, \mathbb{R}^n)$. ∎

We now recall a crucial property for the Palais-Smale condition to hold.

LEMMA 4. *Let (γ^h) be a bounded sequence in $H^1(\mathbb{S}^1, \mathbb{R}^n)$ and set*

$$\left\langle w^h, \eta\right\rangle = \int_0^{2\pi}\sum_{i,j=1}^{n} T^{ij}\left(\gamma^h\right)\dot{\gamma}_i^h\dot{\eta}_j\, d\tau + \frac{1}{2}\int_0^{2\pi}\sum_{i,j=1}^{n} D_s T^{ij}\left(\gamma^h\right)\cdot\eta\dot{\gamma}_i^h\dot{\gamma}_j^h\, d\tau,$$

for all $\eta \in C_c^\infty(\mathbb{S}^1, \mathbb{R}^n)$. Then, if (w^h) is strongly convergent to some w in $H^{-1}(\mathbb{S}^1, \mathbb{R}^n)$, (γ^h) admits a strongly convergent subsequence in $H^1(\mathbb{S}^1, \mathbb{R}^n)$.

PROOF. Since in our setting $L^1(\mathbb{S}^1, \mathbb{R}^n) \subseteq H^{-1}(\mathbb{S}^1, \mathbb{R}^n)$, the proof is standard. ∎

We point out that the previous lemma is absolutely nontrivial in more than one variable. See [1,7].

We now come to one of the main tools of this paper.

THEOREM 3. *There is $M > 0$ such that $\tilde{\mathcal{L}}$ satisfies the $(\mathrm{PS})_c$-condition for $c \geq M$.*

PROOF. Taking into account Lemma 3.3 of [1], combine Lemma 3 and Lemma 4. ∎

4. EXISTENCE OF MULTIPLE PERIODIC ORBITS

Now, let $\{e_1, \dots, e_n\}$ be the standard basis in $\mathbb{R}^n$ and define, for each $1 \le i \le n$,

$$E_{m,i} := \operatorname{span}\{v_{j,k} = \sin(j\tau)\, e_k, w_{j,k} = \cos(j\tau)\, e_k : 1 \le j \le m, 1 \le k \le i\}.$$

By inequality (2.8), there exists $R_{m,i} > 0$ such that

$$\forall\, \gamma \in E_{m,i}, \qquad \|\gamma\|_{1,2} \ge R_{m,i} \Longrightarrow \tilde{\mathfrak{L}}(\gamma) \le 0.$$

If $D_{m,i} = B_{R_{m,i}} \cap E_{m,i}$, we say that $\eta \in C(D_{m,i}, H^1(\mathbb{S}^1, \mathbb{R}^n))$ is equivariant if

$$\forall\, \vartheta \in [0, 2\pi[, \qquad \eta(T_\vartheta(\gamma)) = T_\vartheta \eta(\gamma).$$

Finally, set

$$\Gamma_{k,i} = \left\{\eta \in C\left(D_{m,i}, H^1\right) : \eta \text{ equiv. } \eta(\gamma) = \gamma \text{ if } \|\gamma\| = R_{k,i} \text{ or } \gamma \in E_{0,n}\right\}$$

and

$$b_{k,i} = \inf_{\eta \in \Gamma_{k,i}} \max_{\gamma \in D_{k,i}} \tilde{\mathfrak{L}}\left(\eta\left(\gamma\right)\right).$$

The following result depends on an $\mathbb{S}^1$-version of Borsuk-Ulam's Theorem.

LEMMA 5. *For each $k \in \mathbb{N}$, $1 \le i \le n$, $\varrho \in]0, R_{k,i}[$, and $\eta \in \Gamma_{k,i}$,*

$$\eta(D_{k,i}) \cap \partial B(0, \varrho) \cap E^{\perp}_{k,i-1} \neq \emptyset.$$

PROOF. See [3, Lemma 2.20]. ∎

LEMMA 6. *There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that, for each $1 \le i \le n$,*

$$\forall\, k \ge k_0, \qquad b_{k,i} \ge \beta k^{(\sigma+2)/(\sigma-2)}.$$

PROOF. If $k \ge 1$ and $\gamma \in \partial B(0, \varrho) \cap E^{\perp}_{k,i-1}$, arguing as in [1, Lemma 5.3], by (1.5), we have

$$\tilde{\mathfrak{L}}(\gamma) \ge \frac{\nu}{4}\varrho^2 - \alpha_1 \|\gamma\|^{\sigma}_{\sigma} - \alpha_2 - \alpha_3 \|\gamma\|_2,$$

for some $\alpha_1, \alpha_2, \alpha_3 > 0$. Now, Gagliardo-Niremberg's inequality implies that

$$\|\gamma\|_{\sigma} \le \alpha_4 \|\gamma\|^{(\sigma-2)/2\sigma}_{1,2} \|\gamma\|^{(\sigma+2)/2\sigma}_{2},$$

for some $\alpha_4 > 0$ and all $\gamma \in H^1$. Moreover,

$$\|\gamma\|_2 \le \frac{1}{k} \|\dot{\gamma}\|_2,$$

for each $\gamma \in E^{\perp}_{k,i-1}$. Continuing as in [1, Lemma 5.3], we conclude the proof. ∎

For each $k \in \mathbb{N}$, we now set

$$U_{k,i} = \left\{\gamma = \tau v_{k,i+1} + \phi : \tau \in [0, R_{k,i+1}], \phi \in B\left(0, R_{k,i+1}\right) \cap E_{k,i}, \|\gamma\|_{1,2} \le R_{k,i+1}\right\},$$
$$\Lambda_{k,i} = \left\{\lambda \in C\left(U_{k,i}, H^1\right) : \lambda_{|D_{k,i}} \in \Gamma_{k,i}, \lambda|_{\partial B(0,R_{k,i+1}) \cup ((B(0,R_{k,i+1}) \setminus B(0,R_{k,i})) \cap E_{k,i})} = Id\right\}.$$

We now recall the main existence tool from critical point theory.

THEOREM 4. *Assume that* $c_{k,i} > b_{k,i} \geq M$. *If* $0 < \delta < c_{k,i} - b_{k,i}$ *and*

$$\Lambda_{k,i}(\delta) := \left\{ \lambda \in \Lambda_{k,i} : \tilde{\mathfrak{L}}(\lambda(\gamma)) \leq b_{k,i} + \delta, \text{ for } \gamma \in D_{k,i} \right\},$$

set

$$c_{k,i}(\delta) = \inf_{\lambda \in \Lambda_{k,i}(\delta)} \max_{\gamma \in U_{k,i}} \tilde{\mathfrak{L}}(\lambda(\gamma)).$$

Then $c_{k,i}(\delta)$ *is a critical value for* $\tilde{\mathfrak{L}}$.

PROOF. Argue essentially as in [3, Lemma 2.29]. ∎

It only remains to show that condition $c_{k,i} = b_{k,i}$ for k large is not permitted.

LEMMA 7. *Assume that* $c_{k,i} = b_{k,i}$ *for all* $k \geq k_1$ *and* $1 \leq i \leq n$. *Then, there exist* $\gamma > 0$ *and* $\tilde{k} \geq k_1$ *with*

$$b_{\tilde{k},i} \leq \gamma \tilde{k}^{\mu/(\mu-1)}.$$

PROOF. Choose $k \geq k_1$, $1 \leq i \leq n$, and $\varepsilon > 0$ and let $\lambda \in \Lambda_{k,i}$ be such that

$$\max_{\gamma \in U_{k,i}} \tilde{\mathfrak{L}}(\lambda(\gamma)) \leq b_{k,i} + \varepsilon.$$

Now, let $\tilde{\lambda}(\gamma) = \lambda(\gamma)$ and $\tilde{\lambda}(T_\vartheta \gamma) = T_\vartheta \lambda(\gamma)$ for $\gamma \in U_{k,i}$. It is easy to show that $\tilde{\lambda} \in \Gamma_{k,i+1}$. Then, arguing as in [1, Lemma 5.6], we get

$$b_{k,i+1} \leq b_{k,i} + \beta \left(|b_{k,i}|^{1/\mu} + 1 \right),$$

for $k \geq k_1$. The proof now goes on as in [3, Lemma 2.31]. ∎

Finally, we come to the proof of Theorem 1. Since $\sigma < 4\mu - 2$ implies that $\mu/(\mu - 1) < (\sigma + 2)/(\sigma - 2)$, combining Lemmas 6 and 7, we deduce, by Theorem 4 that $(c_{k,i}(\delta))$ is a sequence of critical values for $\tilde{\mathfrak{L}}$. Whence, by Theorem 4, $\mathfrak{L}$ has a sequence of critical values.

REFERENCES

1. S. Paleari and M. Squassina, A multiplicity result for perturbed symmetric quasilinear elliptic systems, *Adv. Differential Equations* (to appear).
2. A. Bahri and H. Berestyki, Existence d'une infinitè de solutions periodiques systèmes hamiltoniens en prèsence d'un terme de contrainte, *Acad. Sci. Sér. I* **292**, 315–318, (1981).
3. P.H. Rabinowitz, Multiple critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.* **272** (2), 753–769, (1982).
4. A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14**, 349–381, (1973).
5. P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, In *CBMS Reg. Conf. Series Math., Volume 65*, Amer. Math. Soc., Providence, RI, (1986).
6. G. Arioli and F. Gazzola, Existence and multiplicity results for quasilinear elliptic differential systems, *Comm. Partial Differential Equations* (to appear).
7. M. Squassina, Existence of multiple solutions for quasilinear diagonal elliptic systems, *Electron. J. Diff. Eqns.* **14**, 1–12, (1999).
8. M. Struwe, Quasilinear elliptic eigenvalue problems, *Comment. Math. Helvetici* **58**, 509–527, (1983).