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# ON THE EXISTENCE OF POSITIVE ENTIRE SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS

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Abstract. Via non-smooth critical point theory, we prove existence of entire positive solutions for a class of nonlinear elliptic problems with asymptotic $p$-Laplacian behaviour and subjected to natural growth conditions.

## 1. Introduction

In the last few years there has been a growing interest in the study of positive solutions to variational quasilinear equations in unbounded domains of $\mathbb{R}^n$, since these problems are involved in various branches of mathematical physics (see [4]).

Since 1988, quasilinear elliptic equations of the form

$$-\mathrm{div}(\varphi(\nabla u)) = g(x,u) \quad \text{in } \mathbb{R}^n \tag{1}$$

have been extensively treated, among the others, in [2], [8], [12], [14], [20] by means of a combination of topological and variational techniques.

Moreover, existence of a positive solution $u \in H^1(\mathbb{R}^n)$ for the more general equation

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u) D_i u D_j u + b(x)u = g(x,u) \quad \text{in } \mathbb{R}^n,$$

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behaving asymptotically ($|x| \to \infty$) like the problem

$$-\Delta u + \lambda u = u^{q-1} \quad \text{in } \mathbb{R}^n,$$

for some suitable $\lambda > 0$ and $q > 2$, has been firstly studied in 1996 in [9] via techniques of non-smooth critical point theory.

On the other hand, more recently, in a bounded domain $\Omega$ of $\mathbb{R}^n$ some existence results for fully nonlinear problems of the type

$$\begin{cases} -\mathrm{div}\,(\nabla_\xi \mathcal{L}(x,u,\nabla u)) + D_s\mathcal{L}(x,u,\nabla u) = g(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

have been established in [1], [17], [18].

The goal of this paper is to prove existence of a nontrivial positive solution in $W^{1,p}(\mathbb{R}^n)$ for the nonlinear elliptic equation

$$-\mathrm{div}\,(\nabla_\xi \mathcal{L}(x,u,\nabla u)) + D_s\mathcal{L}(x,u,\nabla u) + b(x)|u|^{p-2}u = g(x,u) \quad \text{in } \mathbb{R}^n, \tag{3}$$

behaving asymptotically like the $p$-Laplacian problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = u^{q-1} \quad \text{in } \mathbb{R}^n,$$

for some suitable $\lambda > 0$ and $q > p$. In other words, equation (3) tends to regularize as $|x| \to \infty$ together with its associated functional $f : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$

$$f(u) = \int_{\mathbb{R}^n} \mathcal{L}(x,u,\nabla u)\,dx + \frac{1}{p}\int_{\mathbb{R}^n} b(x)|u|^p\,dx - \int_{\mathbb{R}^n} G(x,u)\,dx. \tag{4}$$

Since in general $f$ is continuous but not even locally Lipschitzian, unless $\mathcal{L}$ does not depend on $u$ or the growth conditions on $\mathcal{L}$ are very restrictive, we shall refer to the non-smooth critical point theory developed in [7], [10], [11], [13], [16] and we shall follow the approach of [9].

We assume that $1 < p < n$, the function $\mathcal{L} : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in $x$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class $C^1$ in $(s,\xi)$ for a.e. $x \in \mathbb{R}^n$ and $\mathcal{L}(x,s,\cdot\,)$ is strictly convex and homogeneous of degree $p$. Take $b \in L^\infty(\mathbb{R}^n)$ with $\underline{b} \le b(x) \le \overline{b}$ for a.e. $x \in \mathbb{R}^n$ for some $\underline{b}, \overline{b} > 0$. Moreover, we shall assume that:

($\mathrm{H}_1$) there exists $\nu > 0$ such that

$$\nu|\xi|^p \le \mathcal{L}(x,s,\xi) \le \frac{1}{p}|\xi|^p, \tag{5}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$,

($\mathrm{H}_2$) there exists $c_1 > 0$ such that

$$|D_s\mathcal{L}(x,s,\xi)| \le c_1|\xi|^p, \tag{6}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, there exist $c_2 > 0$ and $a \in L^{p'}(\mathbb{R}^n)$ such that

$$|\nabla_\xi \mathcal{L}(x,s,\xi)| \le a(x) + c_2|s|^{p^*/p'} + c_2|\xi|^{p-1}, \tag{7}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

($H_3$) there exists $R > 0$ such that

$$s \geq R \Rightarrow D_s\mathcal{L}(x, s, \xi)s \geq 0, \tag{8}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

($H_4$) uniformly in $s \in \mathbb{R}$ and $\xi, \eta \in \mathbb{R}^n$ with $|\xi| \leq 1$ and $|\eta| \leq 1$

$$\lim_{|x|\to\infty} \nabla_\xi \mathcal{L}(x, s, \xi) \cdot \eta = |\xi|^{p-2}\xi \cdot \eta, \tag{9}$$

$$\lim_{|x|\to\infty} D_s\mathcal{L}(x, s, \xi)s = 0, \tag{10}$$

$$\lim_{|x|\to\infty} b(x) = \lambda, \tag{11}$$

for some $\lambda > 0$ and with $b(x) \leq \lambda$ for a.e. $x \in \mathbb{R}^n$.

($G_1$) $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $G(x, s) = \int_0^s g(x, t)\, dt$ and there exist $\beta > 0$ and $q > p$ such that

$$s > 0 \Rightarrow 0 < qG(x, s) \leq g(x, s)s, \tag{12}$$

$$(q - p)\mathcal{L}(x, s, \xi) - D_s\mathcal{L}(x, s, \xi)s \geq \beta|\xi|^p, \tag{13}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Moreover, there exist $\sigma \in \,]p, p^*[$ and $c > 0$ such that

$$|g(x, s)| \leq d(x) + c|s|^{\sigma-1}, \tag{14}$$

for a.e. $x \in \mathbb{R}^n$ and all $s > 0$, where $d \in L^r(\mathbb{R}^n)$, $r \in [np'/(n + p'), p'[$.

($G_2$) we assume that

$$\lim_{|x|\to\infty} \frac{g(x, s)}{s^{q-1}} = 1, \tag{15}$$

uniformly in $s > 0$, and

$$\lim_{|s|\to 0} \frac{G(x, s)}{|s|^p} = 0, \tag{16}$$

uniformly in $x \in \mathbb{R}^n$, and $g(x, s) \geq s^{q-1}$ for each $s > 0$.

Under the previous assumptions, the following is our main result.

Theorem 1. *The Euler's equation of $f$*

$$-\mathrm{div}\,(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s\mathcal{L}(x, u, \nabla u) + b|u|^{p-2}u = g(x, u) \quad \text{in } \mathbb{R}^n \tag{17}$$

*admits at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$.*

This result extends to a more general setting Theorem 2 of [9] dealing with the case:

$$\mathcal{L}(x, s, \xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x, s)\xi_i\xi_j,$$

and Theorem 2.1 of [8] involving integrands of the type:

$$\mathcal{L}(x,\xi)=\frac{1}{p}a(x)|\xi|^p,$$

where $a\in L^\infty(\mathbb{R})$ and $1<p<n$. Let us remark that we assume (8) for large values of $s$, while in [9] it was supposed that, for a.e. $x\in\mathbb{R}^n$ and all $\xi\in\mathbb{R}^n$,

$$\forall s\in\mathbb{R}:\ \sum_{i,j=1}^n sD_s a_{ij}(x,s)\xi_i\xi_j\geq 0.$$

This assumption has been widely considered in literature, not only in studying existence but also to ensure local boundedness of weak solutions (see e.g. [1]).

Condition (13) has been already used in [1], [17], [18] and seems to be a natural extension of what happens in the quasilinear case [7].

We point out that in a bounded domain, conditions (12) and (13) may be assumed for large values of $s$ (see e.g. [18]).

Finally (9)–(11) and (15) fix the asymptotic behaviour of (3). By (9) and (10), there exist two maps $\varepsilon_1:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ and $\varepsilon_2:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ such that

$$\nabla_\xi\mathcal{L}(x,s,\xi)\cdot\eta=|\xi|^{p-2}\xi\cdot\eta+\varepsilon_1(x,s,\xi,\eta)|\xi|^{p-1}|\eta|,\tag{18}$$

$$D_s\mathcal{L}(x,s,\xi)s=\varepsilon_2(x,s,\xi)|\xi|^p,\tag{19}$$

where $\varepsilon_1(x,s,\xi,\eta)\to 0$ and $\varepsilon_2(x,s,\xi)\to 0$ as $|x|\to\infty$ uniformly in $s\in\mathbb{R}$ and $\xi,\eta\in\mathbb{R}^n$.

## 2. Recalls from non-smooth critical point theory

We recall from [7] two basic definitions in a general setting.

Definition 1. Let $(X,d)$ be a metric space, $f:X\to\mathbb{R}$ a continuous function and $u\in X$. We denote by $|df|(u)$ the supremum of $\sigma\in[0,\infty[$ such that there exist $\delta>0$ and a continuous map $\mathcal{H}:B_\delta(u)\times[0,\delta]\to X$ such that, for all $(v,t)\in B_\delta(u)\times[0,\delta]$,

$$d(\mathcal{H}(v,t),v)\leq t,\quad f(\mathcal{H}(v,t))\leq f(v)-\sigma t.$$

We say that the extended real number $|df|(u)$ is the *weak slope* of $f$ at $u$.

Definition 2. Let $(X,d)$ be a metric space, $f:X\to\mathbb{R}$ a continuous function and $u\in X$. We say that $u$ is a *critical point* of $f$ if $|df|(u)=0$.

We now introduce the following variant of the classical $(\mathrm{PS})_c$ condition.

DEFINITION 3. Let $c \in \mathbb{R}$. A sequence $(u_h) \subset W^{1,p}(\mathbb{R}^n)$ is said to be a *concrete Palais–Smale sequence at level* $c$ ($(\mathrm{CPS})_c$*-sequence*, in short) for $f$, if $f(u_h) \to c$,

$$-\mathrm{div}\,(\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)) + D_s\mathcal{L}(x, u_h, \nabla u_h) \in W^{-1,p'}(\mathbb{R}^n)$$

eventually, as $h \to \infty$ and

$$-\mathrm{div}\,(\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)) + D_s\mathcal{L}(x, u_h, \nabla u_h) + b(x)|u_h|^{p-2}u_h - g(x, u_h) \to 0$$

strongly in $W^{-1,p'}(\mathbb{R}^n)$. We say that $f$ satisfies the *concrete Palais–Smale condition at level* $c$ ($(\mathrm{CPS})_c$ in short), if every $(\mathrm{CPS})_c$-sequence for $f$ admits a strongly convergent subsequence.

The following proposition connects the abstract framework of non-smooth critical point theory with the weak solutions of our problem.

PROPOSITION 1. *The functional $f$ is continuous and if $|df|(u) < \infty$ it results*

$$-\mathrm{div}\,(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s\mathcal{L}(x, u, \nabla u) + b|u|^{p-2}u - g(x, u) \in W^{-1,p'}(\mathbb{R}^n)$$

*and*

$$\| - \mathrm{div}\,(\nabla_\xi \mathcal{L}(x, u, \nabla u)) + D_s\mathcal{L}(x, u, \nabla u) + b|u|^{p-2}u - g(x, u)\|_{-1,p'} \le |df|(u).$$

PROOF. See [7, Theorem 2.1.3]. □

As a consequence, each critical point of $f$ solves (17) in the sense of distributions.

## 3. The concrete Palais–Smale condition

Let us now set, for a.e. $x \in \mathbb{R}^n$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$\widetilde{\mathcal{L}}(x, s, \xi) = \begin{cases} \mathcal{L}(x, s, \xi) & \text{if } s \ge 0, \\ \mathcal{L}(x, 0, \xi) & \text{if } s < 0, \end{cases} \qquad \widetilde{g}(x, s) = \begin{cases} g(x, s) & \text{if } s \ge 0, \\ 0 & \text{if } s < 0. \end{cases} \tag{20}$$

We define a modified functional $\widetilde{f} : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ setting

$$\widetilde{f}(u) = \int_{\mathbb{R}^n} \widetilde{\mathcal{L}}(x, u, \nabla u)\, dx + \frac{1}{p}\int_{\mathbb{R}^n} b(x)|u|^p\, dx - \int_{\mathbb{R}^n} \widetilde{G}(x, u)\, dx. \tag{21}$$

Then the Euler's equation of $\widetilde{f}$ is given by

$$-\mathrm{div}\,(\nabla_\xi \widetilde{\mathcal{L}}(x, u, \nabla u)) + D_s\widetilde{\mathcal{L}}(x, u, \nabla u) + b(x)|u|^{p-2}u = \widetilde{g}(x, u) \quad \text{in } \mathbb{R}^n. \tag{22}$$

Lemma 1. *If $u \in W^{1,p}(\mathbb{R}^n)$ is a solution of* (22), *then $u$ is a positive solution of* (17).

Proof. Let $Q : \mathbb{R} \to \mathbb{R}$ the Lipschitz map defined by

$$Q(s) = \begin{cases} 0 & \text{if } s \geq 0, \\ s & \text{if } -1 \leq s \leq 0, \\ -1 & \text{if } s \leq -1. \end{cases}$$

Testing $\widetilde{f}'(u)$ with $Q(u) \in W^{1,p} \cap L^\infty(\mathbb{R}^n)$ and taking into account (20) we have

$$\begin{aligned} 0 &= \widetilde{f}'(u)(Q(u)) \\ &= \int_{\mathbb{R}^n} \nabla_\xi \widetilde{\mathcal{L}}(x,u,\nabla u) \cdot \nabla Q(u)\,dx + \int_{\mathbb{R}^n} D_s \widetilde{\mathcal{L}}(x,u,\nabla u) Q(u)\,dx \\ &\quad + \int_{\mathbb{R}^n} b(x)|u|^{p-2} u Q(u)\,dx - \int_{\mathbb{R}^n} \widetilde{g}(x,u) Q(u)\,dx \\ &= \int_{\{-1<u<0\}} \nabla_\xi \mathcal{L}(x,0,\nabla u) \cdot \nabla u\,dx + \int_{\{u<0\}} D_s \widetilde{\mathcal{L}}(x,u,\nabla u) Q(u)\,dx \\ &\quad + \int_{\mathbb{R}^n} b(x)|u|^{p-2} u Q(u)\,dx - \int_{\{u<0\}} \widetilde{g}(x,u) Q(u)\,dx \\ &= \int_{\{-1<u<0\}} p\mathcal{L}(x,0,\nabla u)\,dx + \int_{\mathbb{R}^n} b(x)|u|^{p-2} u Q(u)\,dx \\ &\geq \underline{b} \int_{\mathbb{R}^n} |u|^{p-2} u Q(u)\,dx \geq 0. \end{aligned}$$

In particular, it results $Q(u) = 0$, namely $u \geq 0$. □

Therefore, without loss of generality, we shall suppose that

$$g(x,s) = 0, \quad \mathcal{L}(x,s,\xi) = \mathcal{L}(x,0,\xi) \quad \text{for all } s \leq 0,$$

for a.e. $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$.

Lemma 2. *Let $c \in \mathbb{R}$. Then each* $(\mathrm{CPS})_c$*-sequence for $f$ is bounded in* $W^{1,p}(\mathbb{R}^n)$.

Proof. If $(u_h)$ is a $(\mathrm{CPS})_c$-sequence for $f$, arguing as in [9, Lemma 2], since

$$f(u_h) - \frac{1}{q} f'(u_h)(u_h) = c + o(1)$$

as $h \to \infty$, by (12) and (13) we get

$$\beta \int_{\mathbb{R}^n} |\nabla u_h|^p\,dx + \frac{q-p}{p} \underline{b} \int_{\mathbb{R}^n} |u_h|^p\,dx \leq C, \tag{23}$$

for some $C > 0$, hence the assertion. □

Let us note that there exists $M > 0$ such that

$$|D_s \mathcal{L}(x,s,\xi)| \leq M \nabla_\xi \mathcal{L}(x,s,\xi) \cdot \xi \tag{24}$$

for a.e. $x \in \mathbb{R}^n$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

We now prove a local compactness property for $(\mathrm{CPS})_c$-sequences. In the following, $\Omega \Subset \mathbb{R}^n$ will always denote an open and bounded subset of $\mathbb{R}^n$.

Theorem 2. *Let $(u_h)$ be a bounded sequence in $W^{1,p}(\mathbb{R}^n)$ and for each $v \in C_c^\infty(\mathbb{R}^n)$ set*

$$\langle w_h, v\rangle = \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla v\, dx + \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) v\, dx. \tag{25}$$

*If $(w_h)$ is strongly convergent to some $w$ in $W^{-1,p'}(\Omega)$ for each $\Omega \Subset \mathbb{R}^n$, then $(u_h)$ admits a strongly convergent subsequence in $W^{1,p}(\Omega)$ for each $\Omega \Subset \mathbb{R}^n$.*

Proof. Since $(u_h)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, we find $u$ in $W^{1,p}(\mathbb{R}^n)$ such that, up to a subsequence, $u_h \rightharpoonup u$ in $W^{1,p}(\mathbb{R}^n)$. Moreover, for each $\Omega \Subset \mathbb{R}^n$, we have

$$u_h \to u \quad \text{in } L^p(\Omega), \qquad u_h(x) \to u(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

By a natural extension of [5, Theorem 2.1] to unbounded domains, we have

$$\nabla u_h(x) \to \nabla u(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Then, following the blueprint of [18, Theorem 3.4], we obtain for each $v \in C_c^\infty(\mathbb{R}^n)$

$$\langle w, v\rangle = \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla v\, dx + \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u, \nabla u) v\, dx. \tag{26}$$

Choose now $\Omega \Subset \mathbb{R}^n$ and fix a positive smooth cut-off function $\eta$ on $\mathbb{R}^n$ with $\eta = 1$ on $\Omega$. Moreover, let $\vartheta : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\vartheta(s) = \begin{cases} Ms & \text{if } 0 < s < R, \\ MR & \text{if } s \geq R, \\ -Ms & \text{if } -R < s < 0, \\ MR & \text{if } s \leq -R, \end{cases} \tag{27}$$

where $M$ is as in (24). Since by [18, Proposition 3.1] $v_h = \eta u_h \exp\{\vartheta(u_h)\}$ are admissible test functions for (25), we get

$$\begin{aligned} &\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \eta \exp\{\vartheta(u_h)\}\, dx - \langle w_h, \eta u_h \exp\{\vartheta(u_h)\}\rangle \\ &+ \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \eta u_h \exp\{\vartheta(u_h)\}\, dx \\ &+ \int_{\mathbb{R}^n} [D_s \mathcal{L}(x, u_h, \nabla u_h) + \vartheta'(u_h) \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h]\, \eta u_h \exp\{\vartheta(u_h)\}\, dx = 0. \end{aligned}$$

Let us observe that

$$\nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \to \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Since, for each $h \in \mathbb{N}$, we have

$$[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \vartheta'(u_h)\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h]\eta u_h \exp\{\vartheta(u_h)\} \leq 0,$$

Fatou's Lemma yields:

$$\begin{aligned}\limsup_h \int_{\mathbb{R}^n} &[-D_s\mathcal{L}(x,u_h,\nabla u_h) \\ &\quad - \vartheta'(u_h)\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h]\cdot\eta u_h \exp\{\vartheta(u_h)\}\,dx \\ &\leq \int_{\mathbb{R}^n}[-D_s\mathcal{L}(x,u,\nabla u) - \vartheta'(u)\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u]\,\eta u\exp\{\vartheta(u)\}\,dx.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}\limsup_h &\int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\eta\exp\{\vartheta(u_h)\}\,dx \\ &= \limsup_h\Big\{\int_{\mathbb{R}^n}[-D_s\mathcal{L}(x,u_h,\nabla u_h) - \vartheta'(u_h)\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h] \\ &\quad\cdot\eta u_h\exp\{\vartheta(u_h)\}\,dx + \langle w_h, \eta u_h\exp\{\vartheta(u_h)\}\rangle \\ &\quad - \int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\eta u_h\exp\{\vartheta(u_h)\}\,dx\Big\} \\ &\leq \Big\{\int_{\mathbb{R}^n}[-D_s\mathcal{L}(x,u,\nabla u) - \vartheta'(u)\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u]\eta u\exp\{\vartheta(u)\}\,dx \\ &\quad + \langle w, \eta u\exp\{\vartheta(u)\}\rangle - \int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla\eta u\exp\{\vartheta(u)\}\,dx\Big\} \\ &= \int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u\eta\exp\{\vartheta(u)\}\,dx,\end{aligned}$$

where we used (26) with $v = \eta u\exp\{\vartheta(u)\}$. In particular, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u\eta\exp\{\vartheta(u)\}\,dx \qquad (28)\\ &\qquad\leq \liminf_h\int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\eta\exp\{\vartheta(u_h)\}\,dx \\ &\qquad\leq \limsup_h\int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\eta\exp\{\vartheta(u_h)\}\,dx \\ &\qquad\leq \int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u\,\eta\exp\{\vartheta(u)\}\,dx, \qquad (29)\end{aligned}$$

namely

$$\begin{aligned}\lim_h\int_{\mathbb{R}^n}&\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h\eta\exp\{\vartheta(u_h)\}\,dx \\ &= \int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u,\nabla u)\cdot\nabla u\eta\exp\{\vartheta(u)\}\,dx. \qquad (30)\end{aligned}$$

Since $\mathcal{L}(x,s,\cdot\,)$ is $p$-homogeneous by (5) for each $h \in \mathbb{N}$ we have

$$\nu\eta p|\nabla u_h|^p \leq \eta\exp\{\vartheta(u_h)\}\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla u_h.$$

By the generalized Lebesgue's theorem we deduce that

$$\lim_h \int_{\mathbb{R}^n} \eta |\nabla u_h|^p \, dx = \int_{\mathbb{R}^n} \eta |\nabla u|^p \, dx.$$

Up to substituting $\eta$ with $\eta^p$, we get

$$\lim_h \int_{\mathbb{R}^n} |\eta \nabla u_h|^p \, dx = \int_{\mathbb{R}^n} |\eta \nabla u|^p \, dx,$$

which implies that $\eta \nabla u_h \to \eta \nabla u$ in $L^p(\mathbb{R}^n)$, namely $\nabla u_h \to \nabla u$ in $L^p(\Omega)$. □

Let us remark that, in general, since the imbedding

$$W^{1,p}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$$

is not compact, we cannot have strong convergence of $(\mathrm{CPS})_c$ sequences on unbounded domains of $\mathbb{R}^n$. Nevertheless, we have the following result.

Lemma 3. *Assume that $(u_h)$ is a $(\mathrm{CPS})_c$-sequence for $f$. Then there exists $u$ in $W^{1,p}(\mathbb{R}^n)$ such that, up to a subsequence, the following facts hold*

(a) *$(u_h)$ converges to $u$ weakly in $W^{1,p}(\mathbb{R}^n)$,*
(b) *$(u_h)$ converges to $u$ strongly in $W^{1,p}(\Omega)$ for each $\Omega \Subset \mathbb{R}^n$,*
(c) *$u$ is a positive weak solution to* (3).

Proof. Since the sequence $(u_h)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, by Lemma 2, of course (a) holds. Now, for fixed $\Omega \Subset \mathbb{R}^n$ we set

$$w_h = \gamma_h + g(x, u_h) - b|u_h|^{p-2}u_h \in W^{-1,p'}(\Omega), \qquad \gamma_h \to 0 \quad \text{in } W^{-1,p'}(\Omega).$$

Then (b) follows by Theorem 2 with $w = g(x,u) - b|u|^{p-2}u$. Finally, by Lemma 1, (c) is a consequence of equation (26). □

Let us now prove a technical lemma that we shall use later.

Lemma 4. *Let $c \in \mathbb{R}$ and $(u_h)$ be a bounded $(\mathrm{CPS})_c$-sequence for $f$. Then for each $\varepsilon > 0$ there exists $\varrho > 0$ such that*

$$\int_{\{|u_h| \le \varrho\}} |\nabla u_h|^p \, dx \le \varepsilon \quad \textit{for each } h \in \mathbb{N}.$$

Proof. Let $\varepsilon, \varrho > 0$ and define, for $\delta \in ]0,1[$, the function $\vartheta_\delta : \mathbb{R} \to \mathbb{R}$ setting

$$\vartheta_\delta(s) = \begin{cases} s & \text{if } |s| \le \varrho, \\ \varrho + \delta\varrho - \delta s & \text{if } \varrho < s < \varrho + \varrho/\delta, \\ -\varrho - \delta\varrho - \delta s & \text{if } -\varrho - \varrho/\delta < s < -\varrho, \\ 0 & \text{if } |s| \ge \varrho + \varrho/\delta. \end{cases} \tag{31}$$

Since $\vartheta_\delta(u_h) \in W^{1,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, we get

$$\begin{aligned}\langle w_h, \vartheta_\delta(u_h)\rangle = &\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h)\, dx \\ &+ \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) \vartheta_\delta(u_h)\, dx \\ &+ \int_{\mathbb{R}^n} b|u_h|^{p-2} u_h \vartheta_\delta(u_h) - \int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h)\, dx.\end{aligned}$$

Then condition (6), $b(x) > 0$ and $|\vartheta_\delta(u_h)| \le \varrho$ yield

$$\begin{aligned}&\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h)\, dx \\ &\quad \le \int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h)\, dx + \varrho \|u_h\|_{1,p}^p + \frac{1}{p' p^{p'/p} \delta^{p'/p}} \|w_h\|_{-1,p'}^{p'} + \delta \|u_h\|_{1,p}^p.\end{aligned}$$

Since $(u_h)$ is bounded in $W^{1,p}(\mathbb{R}^n)$, there exists $\delta > 0$ such that $\delta\|u_h\|_{1,p}^p \le \varepsilon\nu/8$, and

$$\delta \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\, dx \le \varepsilon\nu/2, \tag{32}$$

uniformly with $h \in \mathbb{N}$ so large that $(1/p' p^{p'/p} \delta^{p'/p})\|w_h\|_{-1,p'}^{p'} \le \varepsilon\nu/8$. Now, since

$$\begin{aligned}\int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h)\, dx &\le \int\limits_{\{|u_h| \le \varrho + \varrho/\delta\}} g(x, u_h) u_h\, dx \\ &\le \|d\|_r \Bigg( \int\limits_{\{|u_h| \le \varrho + \varrho/\delta\}} |u_h|^{r'}\, dx \Bigg)^{1/r'} + c \int\limits_{\{|u_h| \le \varrho + \varrho/\delta\}} |u_h|^\sigma\, dx,\end{aligned}$$

we can find $\varrho > 0$ such that

$$\int_{\mathbb{R}^n} g(x, u_h)\vartheta_\delta(u_h)\, dx \le \varepsilon\nu/8$$

and $\varrho\|u_h\|_{1,p}^p \le \varepsilon\nu/8$. Therefore we obtain

$$\int\limits_{\{|u_h| \le \varrho + \varrho/\delta\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla \vartheta_\delta(u_h)\, dx \le \varepsilon\nu/2,$$

namely, taking into account (32),

$$\int\limits_{\{|u_h| \le \varrho\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h\, dx \le \varepsilon\nu.$$

By (5) the proof is complete. □

Let us now introduce the "asymptotic functional" $f_\infty : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ by setting

$$f_\infty(u) = \frac{1}{p}\int_{\mathbb{R}^n} |\nabla u|^p\, dx + \frac{\lambda}{p}\int_{\mathbb{R}^n} |u|^p\, dx - \frac{1}{q}\int_{\mathbb{R}^n} |u^+|^q\, dx$$

and consider the associated $p$-Laplacian problem

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda |u|^{p-2}u = u^{q-1} \quad \text{in } \mathbb{R}^n.$$

(See [8] for the case $p > 2$ and [3] for the case $p = 2$.) We now investigate the behaviour of the functional $f$ over its $(\mathrm{CPS})_c$-sequences.

Lemma 5. *Let $(u_h)$ be a $(\mathrm{CPS})_c$-sequence for $f$ and $u$ its weak limit. Then*

$$f(u_h) \approx f(u) + f_\infty(u_h - u), \tag{33}$$

$$f'(u_h)(u_h) \approx f'(u)(u) + f'_\infty(u_h - u)(u_h - u) \tag{34}$$

*as $h \to \infty$, where the notation $A_h \approx B_h$ means $A_h - B_h \to 0$.*

Proof. By [6, Lemma 2.2] we have the splitting

$$\int_{\mathbb{R}^n} G(x, u_h)\, dx - \int_{\mathbb{R}^n} G(x, u)\, dx - \frac{1}{q}\int_{\mathbb{R}^n} |(u_h - u)^+|^q\, dx = o(1),$$

as $h \to \infty$. Moreover, we easily get

$$\int_{\mathbb{R}^n} b|u_h|^p\, dx - \int_{\mathbb{R}^n} b|u|^p\, dx - \lambda\int_{\mathbb{R}^n} |u_h - u|^p\, dx = o(1),$$

as $h \to \infty$. Observe now that thanks to (18) we have

$$\int\limits_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)\cdot \nabla u_h\, dx - \int\limits_{\{|x|>\varrho\}} |\nabla u_h|^p\, dx \to 0,$$

as $\varrho \to \infty$, uniformly in $h \in \mathbb{N}$ and

$$\int\limits_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x, u, \nabla u)\cdot \nabla u\, dx - \int\limits_{\{|x|>\varrho\}} |\nabla u|^p\, dx \to 0,$$

as $\varrho \to \infty$. Therefore, taking into account that for each $\sigma > 0$ there esists $c_\sigma > 0$ with

$$|\nabla u_h|^p \le c_\sigma |\nabla u|^p + (1+\sigma)|\nabla u_h - \nabla u|^p,$$

we deduce that for each $\varepsilon > 0$ there exists $\varrho > 0$ such that for each $h \in \mathbb{N}$

$$\begin{aligned}\int\limits_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)\cdot \nabla u_h\, dx - \int\limits_{\{|x|>\varrho\}} \nabla_\xi \mathcal{L}(x, u, \nabla u)\cdot \nabla u\, dx \\ - \int\limits_{\{|x|>\varrho\}} |\nabla(u_h - u)|^p\, dx < \widetilde{c}\varepsilon,\end{aligned}$$

for some $\widetilde{c} > 0$. On the other hand, by Lemma 3, $\nabla u_h \to \nabla u$ in $L^p(B(0,\varrho), \mathbb{R}^n)$. Since we deduce

$$\int\limits_{\{|x|\le\varrho\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h)\cdot \nabla u_h\, dx = \int\limits_{\{|x|\le\varrho\}} \nabla_\xi \mathcal{L}(x, u, \nabla u)\cdot \nabla u\, dx + o(1),$$

as $h \to \infty$. Then, for each $\varepsilon > 0$, there exists $\overline{h} \in \mathbb{N}$ such that

$$\int_{\{|x|\leq\varrho\}} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx - \int_{\{|x|\leq\varrho\}} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx - \int_{\{|x|\leq\varrho\}} |\nabla(u_h - u)|^p \, dx < \widehat{c}\varepsilon,$$

for each $h \geq \overline{h}$ and some $\widehat{c} > 0$. Putting the previous inequalities together, we have

$$\int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx = \int_{\mathbb{R}^n} \nabla_\xi \mathcal{L}(x, u, \nabla u) \cdot \nabla u \, dx + \int_{\mathbb{R}^n} |\nabla(u_h - u)|^p \, dx + o(1)$$

as $h \to \infty$. Taking into account that $\mathcal{L}(x, s, \cdot\,)$ is homogeneous of degree $p$, (33) is proved. To prove (34), by the previous step and condition (15), it suffices to show that

$$\int_{\mathbb{R}^n} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = \int_{\mathbb{R}^n} D_s \mathcal{L}(x, u, \nabla u) u \, dx + o(1), \tag{35}$$

as $h \to \infty$. By (19), we find $b_1, b_2 > 0$ such that for each $\varepsilon > 0$ there exists $\varrho > 0$ with

$$\int_{\{|x|>\varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx \leq b_1 \varepsilon, \qquad \int_{\{|x|>\varrho\}} D_s \mathcal{L}(x, u, \nabla u) u \, dx \leq b_2 \varepsilon,$$

uniformly in $h \in \mathbb{N}$. On the other hand, combining (b) of Lemma 3 with (13), the generalized Lebesgue's Theorem yields

$$\int_{\{|x|\leq\varrho\}} D_s \mathcal{L}(x, u_h, \nabla u_h) u_h \, dx = \int_{\{|x|\leq\varrho\}} D_s \mathcal{L}(x, u, \nabla u) u \, dx + o(1),$$

as $h \to \infty$. Then (34) follows by the arbitrariness of $\varepsilon$. □

Let us recall from [15] the following result:

Lemma 6. *Let $1 < p \leq \infty$ and $1 \leq q < \infty$ with $q \neq p^*$. Assume that $(u_h)$ is a bounded sequence in $L^q(\mathbb{R}^n)$ with $(\nabla u_h)$ bounded in $L^p(\mathbb{R}^n)$ and there exists $R > 0$ such that*

$$\sup_{y \in \mathbb{R}^n} \int_{y+B_R} |u_h|^q \, dx = o(1),$$

*as $h \to \infty$. Then $u_h \to 0$ in $L^\alpha(\mathbb{R}^n)$ for each $\alpha \in \,]q, p^*[$.*

Proof. See [15, Lemma I.1]. □

Let $(u_h)$ denote a concrete Palais–Smale sequence for $f$ and let us assume that its weak limit $u$ is 0. If $np'/(n+p') < r < p'$, recalling that by (35) it results

$$\int_{\mathbb{R}^n} D_s\mathcal{L}(x,u_h,\nabla u_h)u_h\,dx = o(1),$$

as $h\to\infty$, we get

$$\begin{aligned} pc = pf(u_h) - f'(u_h)(u_h) + o(1) &\le \int_{\mathbb{R}^n} g(x,u_h)u_h\,dx + o(1) \\ &\le \|d\|_r\|u_h\|_{r'} + c\|u_h\|_\sigma^\sigma + o(1). \end{aligned}$$

Hence, either $\|u_h\|_{r'}$ or $\|u_h\|_\sigma$ does not converge strongly to 0. If we now apply Lemma 6 with $p=q$ (note also that $p<r'$, $\sigma<p^*$), taking into account that $(u_h)$ is bounded in $W^{1,p}(\mathbb{R}^n)$ we find $C>0$ and a sequence $(y_h)\subset\mathbb{R}^n$ with $|y_h|\to\infty$ such that

$$\int_{y_h+B_R} |u_h|^p\,dx \ge C,$$

for some $R>0$. In particular, if $\tau_h u_h(x) = u_h(x-y_h)$, we have

$$\int_{B_R} |\tau_h u_h|^p\,dx \ge C$$

and there exists $\overline{u}\not\equiv 0$ such that

$$\tau_h u_h \rightharpoonup \overline{u} \quad \text{in } W^{1,p}(\mathbb{R}^n). \tag{36}$$

If $r = np'/(n+p')$, the same can be obtained in a similar fashion since for each $\varepsilon>0$ there exist

$$d_{1,\varepsilon}\in L^\ell(\mathbb{R}^n), \quad \ell\in \left]\frac{np'}{n+p'}, p'\right[, \quad d_{2,\varepsilon}\in L^{np'/(n+p')}(\mathbb{R}^n)$$

such that $d = d_{1,\varepsilon}+d_{2,\varepsilon}$ and $\|d_{2,\varepsilon}\|_{np'/(n+p')}\le\varepsilon$. We now show that $\overline{u}$ is a weak solution of:

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = u^{q-1} \quad \text{in } \mathbb{R}^n. \tag{37}$$

Lemma 7. *Let $(u_h)$ a $(\mathrm{CPS})_c$-sequence for $f$ with $u_h\rightharpoonup 0$. Then $\overline{u}$ is a weak solution of* (37)*. Moreover, $\overline{u}>0$.*

Proof. For all $\varphi\in C_c^\infty(\mathbb{R}^n)$ and $h\in\mathbb{N}$ we set $(\tau^h\varphi)(x) := \varphi(x+y_h)$ for all $x\in\mathbb{R}^n$. Since $(u_h)$ is a $(\mathrm{CPS})_c$-sequence for $f$, we have that $f'(u_h)(\tau^h\varphi) = o(1)$, for all $\varphi\in C_c^\infty(\mathbb{R}^n)$ namely, as $h\to\infty$,

$$\begin{aligned} \int_{\mathbb{R}^n} \nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\tau^h\varphi\,dx + \int_{\mathbb{R}^n} D_s\mathcal{L}(x,u_h,\nabla u_h)\tau^h\varphi\,dx \\ + \int_{\mathbb{R}^n} b(x)|u_h|^{p-2}u_h\tau^h\varphi\,dx - \int_{\mathbb{R}^n} g(x,u_h)\tau^h\varphi\,dx = o(1). \end{aligned}$$

Of course, as $h \to \infty$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} b(x)|u_h|^{p-2}u_h\tau^h\varphi\,dx &= \int_{\operatorname{supt}\varphi} b(x-y_h)|\tau_h u_h|^{p-2}\tau_h u_h\varphi\,dx\\ &\to \lambda\int_{\mathbb{R}^n}|\overline{u}|^{p-2}\overline{u}\,\varphi\,dx,\\ \int_{\mathbb{R}^n} g(x,u_h)\tau^h\varphi\,dx &= \int_{\operatorname{supt}\varphi} g(x-y_h,\tau_h u_h)\varphi\,dx \to \int_{\mathbb{R}^n}|\overline{u}^+|^{q-1}\varphi\,dx.\end{aligned}$$

Next we have

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla_\xi\mathcal{L}(x,u_h,\nabla u_h)\cdot\nabla\tau^h\varphi\,dx\\ &\quad=\int_{\operatorname{supt}\varphi}\nabla_\xi\mathcal{L}(x-y_h,\tau_h u_h,\nabla\tau_h u_h)\cdot\nabla\varphi\,dx \to \int_{\mathbb{R}^n}|\nabla\overline{u}|^{p-2}\nabla\overline{u}\cdot\nabla\varphi\,dx.\end{aligned}$$

Now, for each $\varepsilon > 0$, Lemma 4 gives a $\varrho > 0$ such that

$$\int_{\mathbb{R}^n} D_s\mathcal{L}(x,u_h,\nabla u_h)\tau^h\varphi\,dx \le \tilde{c}\varepsilon + \int_{\{|u_h|>\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)\tau^h\varphi\,dx.$$

On the other hand, by (10), we have

$$\begin{aligned}&\int_{\{|u_h|>\varrho\}} D_s\mathcal{L}(x,u_h,\nabla u_h)\tau^h\varphi\,dx\\ &\qquad=\int_{\operatorname{supt}\varphi\cap\{|\tau_h u_h|>\varrho\}} D_s\mathcal{L}(x-y_h,\tau_h u_h,\nabla\tau_h u_h)\varphi\,dx = o(1),\end{aligned}$$

as $h \to \infty$. By arbitrariness of $\varepsilon$ we conclude the proof. Finally $\overline{u} \ge 0$ follows by Lemma 1 and $\overline{u} > 0$ follows by [19, Theorem 1.1]. □

LEMMA 8. *Let $(u_h)$ be a $(\mathrm{CPS})_c$-sequence for $f$ with $u_h \rightharpoonup 0$. Then*

$$f_\infty(\overline{u}) \le \liminf_h f_\infty(\tau_h u_h).$$

PROOF. Since $(u_h)$ weakly goes to 0, Lemma 5 gives $f'_\infty(u_h)(u_h) \to 0$ as $h \to \infty$, so that $f'_\infty(\tau_h u_h)(\tau_h u_h) \to 0$ as $h \to \infty$, namely

$$\int_{\mathbb{R}^n}|\nabla\tau_h u_h|^p\,dx + \lambda\int_{\mathbb{R}^n}|\tau_h u_h|^p\,dx - \int_{\mathbb{R}^n}(\tau_h u_h^+)^q\,dx \to 0$$

as $h \to \infty$. Therefore

$$f_\infty(\tau_h u_h) - \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^n}(\tau_h u_h^+)^q\,dx \to 0.$$

Similarly, Lemma 7 yields

$$f_\infty(\overline{u}) = \left(\frac{1}{p}-\frac{1}{q}\right)\int_{\mathbb{R}^n}|\overline{u}|^q\,dx,$$

and the assertion follows by Fatou's Lemma. □

Lemma 9. *If $(u_h)$ is a* $(\mathrm{CPS})_c$*-sequence for $f$ with $u_h \rightharpoonup 0$, then $f_\infty(\overline{u}) \le c$.*

Proof. Since Lemma 5 yields $f(u_h) \approx f_\infty(\tau_h u_h)$ as $h \to \infty$, by the previous Lemma we conclude the proof. □

We finally come to the proof of the main result of this paper.

Proof of Theorem 1. Since $G$ is superlinear at $\infty$ (12), we have

$$\forall u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}: \ u \ge 0 \Rightarrow \lim_{t\to\infty} f(tu) = -\infty.$$

Let $v \in C_c^\infty(\mathbb{R}^n)$ positive be such that $f(tv) < 0$ for all $t > 1$ and define the minimax class

$$\Gamma = \{\gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \gamma(0) = 0, \ \gamma(1) = v\},$$

and the minimax value

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)).$$

Let us remark that, for each $u \in W^{1,p}(\mathbb{R}^n)$,

$$f(u) \ge \nu \|\nabla u\|_p^p + \frac{b}{p}\|u\|_p^p - \int_{\mathbb{R}^n} G(x,u)\,dx.$$

Then, by (16), it results

$$\lim_h \frac{\int_{\mathbb{R}^n} G(x,w_h)}{\|w_h\|_{1,p}^p} = 0$$

for each $(w_h)$ that goes to 0 in $W^{1,p}(\mathbb{R}^n)$, $f$ has a mountain pass geometry, and by the deformation Lemma of [7] there exists a $(\mathrm{CPS})_c$-sequence $(u_h) \subset W^{1,p}(\mathbb{R}^n)$ for $f$. By Lemma 3 it results that $(u_h)$ converges weakly to a positive weak solution $u$ of (3). Therefore, if $u \ne 0$, we are done. On the other hand, if $u = 0$ let us consider $\overline{u}$. We now prove that $\overline{u}$ is a weak solution to our problem. Since, for each $u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}$, we have

$$u \ge 0 \Rightarrow \lim_{t\to\infty} f_\infty(tu) = -\infty,$$

we find $R > 0$ so large that

$$\forall a, b \ge 0: \ a + b = R \Rightarrow f_\infty(a\overline{u} + bv) < 0.$$

Define the path $\gamma : [0,1] \to W^{1,p}(\mathbb{R}^n)$ by

$$\gamma(t) = \begin{cases} 3Rt\overline{u} & \text{if } t \in [0,1/3], \\ (3t-1)Rv + (2-3t)R\overline{u} & \text{if } t \in [1/3,2/3], \\ (3R+3t-3Rt-2)v & \text{if } t \in [2/3,1]. \end{cases}$$

Of course we have $\gamma \in \Gamma$, $f_\infty(\gamma(t)) < 0$ for each $t \in \,]1/3,1]$ and by [8, Lemma 2.4]

$$\max_{t\in[0,1/3]} f_\infty(\gamma(t)) = f_\infty(\overline{u}).$$

Hence, by Lemma 8 and the assumptions on $\mathcal{L}$ and $g$, we have

$$c \leq \max_{t\in[0,1]} f(\gamma(t)) \leq \max_{t\in[0,1]} f_\infty(\gamma(t)) = f_\infty(\overline{u}) \leq c.$$

Therefore, since $\gamma$ is an optimal path in $\Gamma$, by the non-smooth deformation Lemma of [7], there exists $\overline{t} \in \,]0,1[$ such that $\gamma(\overline{t})$ is a critical point of $f$ at level $c$. Moreover, $\gamma(\overline{t}) = \overline{u}$, otherwise

$$f(\gamma(\overline{t})) \leq f_\infty(\gamma(\overline{t})) < f_\infty(\overline{u}) = c,$$

in contradiction with $f(\gamma(\overline{t})) = c$. Then $\overline{u}$ is a positive solution to (3). □

REMARK 1. Let $1 < p < n$, $q > p$ and $\lambda > 0$. As a by-product of Theorem 1, taking

$$\mathcal{L}(x,s,\xi) = \frac{1}{p}|\xi|^p + \frac{\lambda}{p}|s|^p - \frac{1}{q}|s|^q,$$

we deduce that the problem

$$-\mathrm{div}\,(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^n, \tag{38}$$

has at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$ (see also [8], [20]).

In some sense, Theorem 1 implies that the $\varepsilon$-perturbed problem

$$-\mathrm{div}\,((1+\varepsilon(x,u,\nabla u))|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = |u|^{q-2}u \quad \text{in } \mathbb{R}^n, \tag{39}$$

has at least one nontrivial positive solution $u \in W^{1,p}(\mathbb{R}^n)$.

REMARK 2. By [1, Lemma 1.4] we have a local boundedness property for solutions of problem (3), namely, for each $\Omega \Subset \mathbb{R}^n$ each weak solution $u \in W^{1,p}(\Omega)$ of (3) belongs to $L^\infty(\Omega)$ provided that in (14) is $d \in L^s(\Omega)$ for a sufficiently large $s$ (see [1], [7]).

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