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Infinitely many solutions for polyharmonic elliptic problems with broken symmetries

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By means of a perturbation argument devised by P. Bolle, we prove the existence of infinitely many solutions for perturbed symmetric polyharmonic problems with non–homogeneous Dirichlet boundary conditions. An extension to the higher order case of the estimate from below for the critical values due to K. Tanaka is obtained.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with $n > 2K$, $K \geqslant 1$,

$$\phi_j \in H^{K-j-\frac{1}{2}}(\partial\Omega), \quad j = 0, \dots, K-1$$

and φ a function in $L^2(\Omega)$. Moreover let

$$2 < \sigma < K_*, \quad K_* = \frac{2n}{n-2K},$$

being K_* the critical Sobolev exponent for the embedding $H_0^K(\Omega) \hookrightarrow L^{K_*}(\Omega)$. The main goal of this paper is to study the existence of multiple solutions for the following polyharmonic elliptic problem

$$(-\Delta)^K u = |u|^{\sigma-2} u + \varphi \quad \text{in} \quad \Omega \tag{P_φ^K}$$

with non–homogeneous Dirichlet boundary conditions

$$\left(\frac{\partial}{\partial\nu}\right)^j u\bigg|_{\partial\Omega} = \phi_j, \quad j = 0, \dots, K-1, \tag{D_ϕ}$$

where ν denotes the outer unit normal to $\partial\Omega$.

So far, many papers have been written on the existence and multiplicity of solutions for second order elliptic problems with Dirichlet boundary conditions, especially by means of variational methods. In particular, if $\Omega \subset \mathbb{R}^n$ ($n \geqslant 3$) is a smooth bounded domain, $\varphi \in L^2(\Omega)$ and $2 < \sigma < \frac{2n}{n-2}$, the following model problem

$$\begin{cases} -\Delta u = |u|^{\sigma-2} u + \varphi & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

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has exercized many researchers in the last decades.

If $\varphi = 0$, the problem is symmetric and multiplicity results come from the equivariant Lusternik–Schnirelmann theory (see [23] and references therein). On the contrary, if $\varphi \neq 0$ the symmetry of the problem breaks down and a natural question is whether the multiplicity persists under perturbation of the odd equation. Partial answers have been given in [1, 2, 3, 9, 19, 22, 24] where existence of infinitely many solutions was obtained via techniques of classical critical point theory, provided that a suitable restriction on the exponent σ is assumed. See also [18] and [21] for some recent extensions to the quasilinear case by means of techniques of nonsmooth critical point theory. The problem of whether (1.1) has an infinite number of solutions for σ all the way up to $\frac{2n}{n-2}$ is still open. For a subset of φ dense in $L^2(\Omega)$, a positive answer was given by Bahri and Lions in [3].

The success in looking for solutions of the non–symmetric problem (1.1) made quite interesting to study the problem

$$\begin{cases} -\Delta u \;=\; |u|^{\sigma-2}\, u + \varphi & \text{in} \quad \Omega\,, \\ \;u \;=\; \phi & \text{on} \quad \partial\Omega\,, \end{cases} \tag{1.2}$$

where, in general, the boundary condition $\phi \in H^{1/2}(\partial\Omega)$ is different from zero. This introduces a double loss of symmetry, since the associated functional contains two terms which fail to be even. Some multiplicity results for (1.2) have been proved in [5, 6, 7, 10, 11] provided that suitable restrictions on σ, depending also on the regularity of Ω, are assumed.

Now, a natural question is how far the results known for the second order case extend to boundary value problems of higher order. As for the case $K = 1$, the unperturbed equation $\left(P_0^K\right)$ with homogeneous boundary conditions $\left(\text{i.e. } \phi_j = 0\right)$ admits infinitely many solutions for any $2 < \sigma < K_*$. On the other hand, to the authors' knowledge, no multiplicity result can be found in the literature for polyharmonic elliptic problems with a non–symmetric nonlinearity and non–homogeneous Dirichlet boundary conditions ($K \geqslant 2$, $\phi_j \neq 0$ and $\varphi \neq 0$).

Many situations lead naturally to higher order problems: for instance, in physics the clamped plate equation, which was intensively studied by Grunau and Sweers [13, 14, 15, 16] from the point of view of positivity preservation; in differential geometry the fourth order conformal operator involving Δ_g^2 discovered by Paneitz (see e. g. [8]).

In order to prove the main results on $\left(P_\varphi^K\right)$, we will apply a method due to Bolle et al. [4, 5] for dealing with problems with broken symmetry. The idea is to consider a continuous path of functionals $(\Phi_t)_{0\leqslant t\leqslant 1}$ where Φ_0 is symmetric and Φ_1 is the functional associated to our problem. Then, as t varies, one proves a preservation of minmax critical levels, thus getting critical points also for Φ_1. It is a standard fact that the critical points of Φ_1 correspond to the solutions of the problem.

We point out that in the case $\sigma = K_*$ Bolle's method does not seem to provide infinitely many solutions in presence of broken symmetries. For the critical growth case (with zero boundary data), we refer to [12] for the existence of one solution in a very general framework.

Let $K \geqslant 1$. We endow the Sobolev space $H_0^K(\Omega)$ with the standard scalar product

$$(u,v)_{K,2} \;=\; \begin{cases} \displaystyle\int_\Omega \Delta^m u\, \Delta^m v\, dx & \text{if} \quad K \;=\; 2m\,, \\ \displaystyle\int_\Omega \nabla\Delta^m u\, \nabla\Delta^m v\, dx & \text{if} \quad K \;=\; 2m+1\,. \end{cases} \tag{1.3}$$

Let now $\psi \in H^K \cap L^\infty(\Omega)$ be the solution of

$$\begin{cases} (-\Delta)^K \psi \;=\; 0 \quad \text{in} \quad \Omega\,, \\ \left.\left(\dfrac{\partial}{\partial\nu}\right)^j \psi\right|_{\partial\Omega} \;=\; \phi_j\,, \quad j \;=\; 0\,, \ldots\,, K-1\,. \end{cases} \tag{1.4}$$

Then the problem $\left(P_\varphi^K\right)$ with boundary conditions (D_ϕ) can be reduced to

$$(-\Delta)^K w \;=\; |w+\psi|^{\sigma-2}\,(w+\psi) + \varphi \quad \text{in} \quad \Omega$$

with homogeneous boundary data. We say that $u \in H^K(\Omega)$ is a *solution* of $\left(P^K_\varphi\right)$ with conditions (D_ϕ), if $u = w + \psi$, where $w \in H^K_0(\Omega)$ satisfies

$$\int_\Omega \Delta^m w\, \Delta^m \eta\, dx \;=\; \int_\Omega |w+\psi|^{\sigma-2}(w+\psi)\eta\, dx + \int_\Omega \varphi\,\eta\, dx\,, \qquad K \;=\; 2m\,,$$
$$\int_\Omega \nabla\Delta^m w\, \nabla\Delta^m \eta\, dx \;=\; \int_\Omega |w+\psi|^{\sigma-2}(w+\psi)\eta\, dx + \int_\Omega \varphi\,\eta\, dx\,, \quad K \;=\; 2m+1\,,$$

for all $\eta \in H^K_0(\Omega)$.

We are now ready to state the main results of the paper.

Theorem 1.1 *Let $n > 2K$ and assume that σ satisfies*

$$2 \;<\; \sigma \;<\; 2\,\frac{n+K}{n}\,.$$

Then $\left(P^K_\varphi\right)$ with boundary conditions (D_ϕ) admits infinitely many solutions.

Theorem 1.1 is new also in the case where $\phi_j = 0$ for each $j = 0, \dots, K-1$. In this situation the following stronger result holds.

Theorem 1.2 *Let $n > 2K$ and assume that σ satisfies*

$$2 \;<\; \sigma \;<\; 2\,\frac{n-K}{n-2K}\,.$$

Then $\left(P^K_\varphi\right)$ with homogeneous boundary conditions admits infinitely many solutions.

These results extend the achievements of [5, 6, 7, 10], dealing with second order equations, to higher order elliptic problems.

Remark 1.3 If $\partial\Omega$ is of class $C^{2K,\alpha}$, $\varphi \in C^{0,\alpha}\big(\overline{\Omega}\big)$ and $\phi_j \in C^{2K-j}(\partial\Omega)$, then each solution of $\left(P^K_\varphi\right)$ belongs to $C^{2K,\alpha}\big(\overline{\Omega}\big)$, hence it is classical (see [17]).

Remark 1.4 Let $f \in C^1(\mathbb{R})$ and set $F(s) = \int_0^s f(t)\,dt$. Suppose that

$$|f(s)| \;\leqslant\; C\big(1+|s|^{\sigma-1}\big)\,, \quad |s| \;\geqslant\; R \implies 0 \;<\; \sigma F(s) \;\leqslant f(s)s$$

for some $C, R > 0$ with F invariant with respect to more general groups of symmetry. Then the previous results can be extended to equations

$$(-\Delta)^K u \;=\; f(u) + \varphi \quad \text{in} \quad \Omega\,.$$

See [10] and Remark 2.2.

2 Bolle's method for broken–symmetry problems

In this section we briefly recall the theory devised by Bolle [4] for dealing with problems with broken symmetry. Let X be an infinite dimensional Hilbert space equipped with the norm $\|\cdot\|_X$ and

$$\Phi \;:\; [0,1]\times X \;\longrightarrow \mathbb{R}$$

a C^2 functional. We set $\Phi_\theta = \Phi(\theta,\cdot)$ if $\theta \in [0,1]$ and we denote by $\Phi'_\theta : X \to X$ the Fréchet derivative of Φ_θ. Assume that

$$X \;=\; X_0 \oplus \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}e_k \oplus \ldots$$

where $\dim(X_0) < +\infty$ and $(e_k)_{k\geqslant 1}$ is an orthonormal system in X. Let $R > 0$ and set

$$\mathscr{C} \;=\; \big\{\zeta \in C(X,X) :\; \zeta \text{ is odd and } \zeta(u) = u \text{ if } \|u\|_X \geqslant R\big\}$$

and

$$c_k \;=\; \inf_{\zeta\in\mathscr{C}}\, \sup_{u\in X_k}\, \Phi_0(\zeta(u))\,, \tag{2.1}$$

where $X_k = \mathbb{R}e_1 \oplus \ldots \oplus \mathbb{R}\,e_k$. Moreover, assume that:

$(\mathscr{B}_1)$**:** Φ satisfies the Palais–Smale condition in $[0,1]\times X$, i.e. any sequence (θ_h, u_h) such that $(\Phi(\theta_h, u_h))$ is bounded and $\Phi'_{\theta_h}(u_h) \to 0$ as $h \to +\infty$, converges on some suitable subsequence;

$(\mathscr{B}_2)$**:** for any $b > 0$ there exists $C_b > 0$ such that

$$\left|\frac{\partial}{\partial\theta}\Phi(\theta, u)\right| \leqslant C_b(\|\Phi'_\theta(u)\|_X + 1)(\|u\|_X + 1)$$

for all $(\theta, u) \in [0,1]\times X$ with $|\Phi_\theta(u)| \leqslant b$;

$(\mathscr{B}_3)$**:** there exist two continuous maps $\eta_1, \eta_2 : [0,1]\times\mathbb{R} \to \mathbb{R}$, $\eta_1 \leqslant \eta_2$, which are Lipschitz continuous with respect to the second variable and such that

$$\eta_1(\theta, \Phi_\theta(u)) \leqslant \frac{\partial}{\partial\theta}\Phi(\theta, u) \leqslant \eta_2(\theta, \Phi_\theta(u)) \tag{2.2}$$

at each critical point u of Φ_θ;

$(\mathscr{B}_4)$**:** Φ_0 is even and for each finite dimensional subspace W of X it results

$$\lim_{\substack{u\in W\\ \|u\|_X\to+\infty}} \sup_{\theta\in[0,1]} \Phi(\theta, u) = -\infty\,.$$

Taken $i = 1, 2$, let us denote by $\psi_i : [0,1]\times\mathbb{R} \to \mathbb{R}$ the solutions of the problem

$$\begin{cases} \dfrac{\partial}{\partial\theta}\psi_i(\theta, s) = \eta_i(\theta, \psi_i(\theta, s))\,, \\ \psi_i(0, s) = s\,. \end{cases}$$

Note that $\psi_i(\theta,\cdot)$ are continuous, non–decreasing on $\mathbb{R}$ and $\psi_1 \leqslant \psi_2$. Set

$$\overline{\eta}_1(s) = \sup_{\theta\in[0,1]} \eta_1(\theta, s)\,, \quad \overline{\eta}_2(s) = \sup_{\theta\in[0,1]} \eta_2(\theta, s)\,.$$

In this framework, the following abstract result can be proved.

Theorem 2.1 *Assume that the sequence*

$$\left(\frac{c_{k+1} - c_k}{\overline{\eta}_1(c_{k+1}) + \overline{\eta}_2(c_k) + 1}\right)$$

is unbounded. Then the functional Φ_1 admits a sequence of critical values $(\tilde{c}_k)$ such that $\psi_2(1, c_k) < \psi_1(1, c_{k+1}) \leqslant \tilde{c}_k$ for every $k \in \mathbb{N}$.

Proof. See [4, Theorem 3] and [5, Theorem 2.2]. ☐

Remark 2.2 Let G be a compact Lie group acting orthogonally on X. It has been recently proved by M. Clapp et al. [10] that the previous result holds provided that Φ_0 is G–invariant.

3 Tanaka's theory for even functionals

In this section we recall some notions and results from [24]. Let X be an infinite dimensional separable Hilbert space and let $f : X \to \mathbb{R}$ be a function of class C^2 satisfying the following conditions:

$(\mathscr{T}_1)$**:** f is even with $f(0) = 0$;

$(\mathscr{T}_2)$**:** for any finite dimensional subspaces W of X there exists $R = R(W) > 0$ such that $f(u) < 0$ for every $u \in W$ with $\|u\| \geqslant R$;

($\mathscr{T}_3$)**:** for every $u \in X$ it is

$$f'(u) \;=\; u + K(u)\,,$$

where $f' : X \to X$ denotes the Fréchet derivative of f and $K : X \to X$ is a compact operator.

Moreover we assume that there exists a sequence (X_k) of subspaces of X such that

$$\dim X_k \;=\; k\,, \quad X \;=\; \overline{\bigcup_{k=1}^{\infty} X_k}\,.$$

For every $k \in \mathbb{N}$ let us set $R_k = R(X_k)$, $D_k = X_k \cap \overline{B(0,R_k)}$,

$$\mathscr{C}_k \;=\; \left\{\gamma \in C(D_k, X) :\; \gamma \;\text{ odd and }\; \gamma\big|_{X_k \cap \partial B(0,R_k)} = \mathrm{Id}\right\} \tag{3.1}$$

and

$$b_k \;=\; \inf_{\gamma \in \mathscr{C}_k} \sup_{u \in D_k} f(\gamma(u))\,. \tag{3.2}$$

Let us now recall the following Palais–Smale conditions.

Definition 3.1 We say that:

(a) f satisfies the (PS) condition, if every sequence (u_h) in X with $(f(u_h))$ bounded and $\|f'(u_h)\|_X \to 0$ as $h \to +\infty$ admits a subsequence converging in X;

(b) f satisfies the $(PS)_k$ condition, if every sequence (u_h) in X_k with $(f(u_h))$ bounded and

$$\left\| \left(f\big|_{X_k}\right)'(u_h) \right\|_{X_k} \;\longrightarrow\; 0$$

as $h \to +\infty$ admits a subsequence converging in X_k;

(c) f satisfies the $(PS)_*$ condition, if every sequence (u_k) in X with $u_k \in X_k$, $(f(u_k))$ bounded and

$$\left\| \left(f\big|_{X_k}\right)'(u_k) \right\|_{X_k} \longrightarrow 0$$

as $k \to +\infty$ admits a subsequence converging in X.

Definition 3.2 Let $u \in X$ be a critical point of f. The *large Morse index of f at u*, denoted by $m^*(f,u)$, is the infimum of the codimensions of the subspaces of X where the quadratic form $f''(u)$ is positive definite.

The next result is the main tool for estimating from below the critical values b_k. For the proof see [24, Theorem B].

Theorem 3.3 *Assume that f satisfies $(\mathscr{T}_1)$–$(\mathscr{T}_3)$, (PS), $(PS)_k$ and $(PS)_*$. Then for each $k \geqslant 1$ there exists $u_k \in X$ such that*

$$f(u_k) \;\leqslant\; b_k\,, \quad f'(u_k) \;=\; 0\,, \quad m^*(f,u_k) \;\geqslant\; k\,.$$

4 Application to polyharmonic problems

Let $\psi \in H^K \cap L^\infty(\Omega)$ be again the solution of the problem (1.4). For $\theta \in [0,1]$, let us consider the functional $\Phi_\theta : H_0^K(\Omega) \to \mathbb{R}$,

$$\Phi_\theta(u) \;=\; \frac{1}{2}\,\|u\|_{K,2}^2 - \frac{1}{\sigma}\int_\Omega |u+\theta\psi|^\sigma\,dx - \theta\int_\Omega \varphi u\,dx\,.$$

Note that

$$\text{for all} \quad u \in H_0^K(\Omega) :\;\; \Phi_0(-u) \;=\; \Phi_0(u)$$

and that critical points of Φ_1 are associated with the weak solutions of problem $\left(P_\varphi^K\right)$ with Dirichlet boundary conditions (D_ϕ).

We now show that Φ_θ satisfies condition $(\mathscr{B}_1)$.

Lemma 4.1 *Let* $(\theta_h, u_h) \subset [0,1] \times H_0^K(\Omega)$ *be such that*

$$(\Phi(\theta_h, u_h)) \ \text{is bounded}, \quad \lim_h \Phi'_{\theta_h}(u_h) \ = \ 0 .$$

Then, on some suitable subsequence, (θ_h, u_h) *converges in* $[0,1] \times H_0^K(\Omega)$.

P r o o f. For every $h \geqslant 1$ we have

$$\big(\Phi'_{\theta_h}(u_h), u_h\big)_{K,2} \ = \ \|u_h\|^2_{K,2} - \int_\Omega |u_h + \theta\psi|^{\sigma-2}(u_h+\theta\psi)u_h\,dx - \theta\int_\Omega \varphi u_h\,dx .$$

Since $\big(\Phi'_{\theta_h}(u_h), u_h\big)_{K,2} = o(\|u_h\|_{K,2})$, for a suitable $B > 0$ and $\varrho \in \left]\frac{1}{\sigma}, \frac{1}{2}\right[$ it results

$$\begin{aligned} B + \varrho\,\|u_h\|_{K,2} \ &\geqslant \ \Phi_{\theta_h}(u_h) - \varrho(\Phi'_{\theta_h}(u_h), u_h)_{K,2} \\ &= \ \left(\frac{1}{2} - \varrho\right)\|u_h\|^2_{K,2} + \left(\varrho - \frac{1}{\sigma}\right)\int_\Omega |u_h + \theta_h\psi|^\sigma dx \\ &\quad - \theta_h\varrho\int_\Omega |u_h+\theta_h\psi|^{\sigma-2}(u_h+\theta_h\psi)\psi\,dx + \theta_h(\varrho-1)\int_\Omega \varphi u_h\,dx \end{aligned}$$

as $h \to +\infty$. Then, fixed $\varepsilon \in \left]0, 1 - \frac{1}{\varrho\sigma}\right[$, since

$$|u_h + \theta_h\psi|^\sigma \ \geqslant \ \frac{1}{2^{\sigma-1}}\left(|u_h|^\sigma - |\psi|^\sigma\right),$$

in view of Young's inequality some computations entail

$$B + \varrho\,\|u_h\|_{K,2} \ \geqslant \ \left(\frac{1}{2} - \varrho\right)\|u_h\|^2_{K,2} + \frac{1}{2^{\sigma-1}}\left(\varrho(1-\varepsilon) - \frac{1}{\sigma}\right)\|u_h\|^\sigma_\sigma - a_\varepsilon$$

for some $a_\varepsilon > 0$, which implies the boundedness of (u_h) in $H_0^K(\Omega)$. Since the map

$$H_0^K(\Omega) \ \xrightarrow{\ \Upsilon\ } \ L^{\frac{K_*}{\sigma-1}}(\Omega) \ \xrightarrow{((-\Delta)^K)^{-1}} \ H_0^K(\Omega), \quad \Upsilon(u) = |u+\theta\psi|^{\sigma-2}(u+\theta\psi)$$

is compact, it is a standard fact that (u_h) strongly converges in $H_0^K(\Omega)$. □

In the following result we see that assumption $(\mathscr{B}_2)$ is also fulfilled.

Lemma 4.2 *For each* $b > 0$ *there exists* $C > 0$ *such that*

$$\left|\frac{\partial}{\partial\theta}\Phi(\theta, u)\right| \ \leqslant \ C\big(1 + \|\Phi'_\theta(u)\|_{K,2}\big)\big(1 + \|u\|_{K,2}\big)$$

for all $(\theta, u) \in [0,1] \times H_0^K(\Omega)$ *with* $|\Phi_\theta(u)| \leqslant b$.

P r o o f. Let $b > 0$. Condition $|\Phi_\theta(u)| \leqslant b$ implies that

$$\theta\int_\Omega \varphi u\,dx \ \geqslant \ \frac{\sigma}{2}\,\|u\|^2_{K,2} - \int_\Omega |u+\theta\psi|^\sigma\,dx - (\sigma-1)\theta\int_\Omega \varphi u\,dx - \sigma b \tag{4.1}$$

and for some $c_1, c_2 > 0$

$$\|u + \theta\psi\|^\sigma_\sigma \ \leqslant \ c_1\|u\|^2_{K,2} + c_2 . \tag{4.2}$$

Therefore, by (4.1) we have

$$\begin{aligned} -(\Phi'_\theta(u), u)_{K,2} &= -\,\|u\|^2_{K,2} + \int_\Omega |u+\theta\psi|^{\sigma-2}(u+\theta\psi)u\,dx + \theta\int_\Omega \varphi u\,dx \\ &\geqslant \Big(\frac{\sigma}{2} - 1\Big)\|u\|^2_{K,2} - \int_\Omega |u+\theta\psi|^{\sigma-2}(u+\theta\psi)\theta\psi\,dx - (\sigma-1)\,\theta\int_\Omega \varphi u\,dx - \sigma b . \end{aligned}$$

Taking into account (4.2), by Hölder and Young inequalities for each $\varepsilon > 0$ there exist $c_{1\varepsilon}, c_{2\varepsilon} > 0$ such that

$$\left| \int_\Omega |u + \theta\psi|^{\sigma-2}(u + \theta\psi)\theta\psi \, dx \right| \leqslant \varepsilon \|u\|^2_{K,2} + c_{1\varepsilon} ,$$

$$\left| \int_\Omega \varphi u \, dx \right| \leqslant \varepsilon \|u\|^2_{K,2} + c_{2\varepsilon} .$$

In particular, by choosing ε small enough, one finds $c_3, c_4 > 0$ such that

$$-(\Phi'_\theta(u), u)_{K,2} \geqslant c_3 \|u\|^2_{K,2} - c_4 . \tag{4.3}$$

On the other hand, since

$$\frac{\partial}{\partial\theta}\Phi(\theta, u) = -\int_\Omega |u + \theta\psi|^{\sigma-2}(u + \theta\psi)\psi \, dx - \int_\Omega \varphi u \, dx , \tag{4.4}$$

by (4.2), arguing as above, for each $\varepsilon > 0$ there exists $c_{3\varepsilon} > 0$ such that

$$\left| \frac{\partial}{\partial\theta}\Phi(\theta, u) \right| \leqslant \varepsilon \|u\|^2_{K,2} + c_{3\varepsilon} . \tag{4.5}$$

Hence, the proof follows by (4.3), (4.5) and a suitable choice of ε. □

Finally, we check that also $(\mathscr{B}_3)$ is satisfied.

Lemma 4.3 *Let $\eta_1, \eta_2 : [0, 1] \times \mathbb{R} \to \mathbb{R}$ be functions defined as*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C\big(s^2 + 1\big)^{(\sigma-1)/2\sigma} \tag{4.6}$$

for a suitable $C > 0$. Then (2.2) *holds at each critical point of Φ_θ. Moreover, if*

$$-\eta_1(\theta, s) = \eta_2(\theta, s) = C\big(s^2 + 1\big)^{1/2\sigma} \tag{4.7}$$

the same holds provided that $\phi_j = 0$ for each $j = 0, \dots, K - 1$.

P r o o f. It follows by (4.4) that there exist $b_1, b_2 > 0$ with

$$\left| \frac{\partial}{\partial\theta}\Phi(\theta, u) \right| \leqslant b_1 \|u + \theta\psi\|^{\sigma-1}_\sigma + b_2 ;$$

analogously, with homogeneous boundary data one gets

$$\left| \frac{\partial}{\partial\theta}\Phi(\theta, u) \right| \leqslant b'_1 \|u\|_\sigma .$$

Therefore, since there exists $C' > 0$ such that at each critical point u of Φ_θ

$$\|u + \theta\psi\|^\sigma_\sigma \leqslant C'\big(\Phi^2_\theta(u) + 1\big)^{1/2} ,$$

condition (2.2) is fulfilled with η_1 and η_2 chosen either as in (4.6) or in (4.7). □

5 The growth estimate from below

The main goal of this section is to get a growth estimate from below for the critical values c_k. The technique relies on a combination of Morse theory with some spectral properties of the higher order Schrödinger operator.

Consider the eigenvalue problem related to the higher order Schrödinger operator

$$(-\Delta)^K u + V(x)u = \lambda u \quad \text{in} \quad \mathbb{R}^n ,$$

where $V \in L^{n/2K}(\mathbb{R}^n)$ and let $(\mu_k) \subset \mathbb{R}$ denote the sequence of eigenvalues of the operator $(-\Delta)^K + V(x)$, repeated according to multiplicity.

Lemma 5.1 *Let $n > 2K$ and $V \in L^{n/2K}(\mathbb{R}^n)$. Then there exists $B_{n,K} > 0$ with*

$$\sharp\{k \in \mathbb{N} :\ \mu_k \leqslant 0\} \ \leqslant\ B_{n,K} \int_{\mathbb{R}^n} V^-(x)^{n/2K}\, dx\,, \tag{5.1}$$

where $\sharp$ denotes the cardinality.

Proof. See [20, Theorem 3]. □

The next result establishes the required estimate from below.

Lemma 5.2 *There exists $\alpha > 0$ such that*

$$b_k \ \geqslant\ \alpha k^{2K\sigma/n(\sigma-2)}$$

for each $k \geqslant 1$.

Proof. We want to apply Theorem 3.3 to the functional $\Phi_0 : H_0^K(\Omega) \to \mathbb{R}$,

$$\Phi_0(u) \ =\ \frac{1}{2}\,\|u\|_{K,2}^2 - \frac{1}{\sigma}\,\|u\|_\sigma^\sigma\,.$$

It is straightforward that Φ_0 satisfies $(\mathscr{T}_1)$ and $(\mathscr{T}_2)$. Moreover, since the map

$$H_0^K(\Omega) \xrightarrow{\ \Upsilon\ } L^{\frac{K_*}{\sigma-1}}(\Omega) \xrightarrow{((-\Delta)^K)^{-1}} H_0^K(\Omega)\,, \quad \Upsilon(u) \ =\ |u|^{\sigma-2}u$$

is compact, then $(\mathscr{T}_3)$ is also fulfilled. By Lemma 4.1 it follows that Φ_0 satisfies the (PS) condition and, in a similar way, we obtain also that Φ_0 satisfies the $(PS)_k$ and $(PS)_*$ conditions. Then, by Theorem 3.3 there exists a sequence (u_k) in $H_0^K(\Omega)$ of critical points of Φ_0 such that $\Phi_0(u_k) \leqslant b_k$ and $m^*(\Phi_0, u_k) \geqslant k$.

If (μ_k) is the sequence of the eigenvalues (repeated according to their multiplicity) of the operator $\{v \mapsto (-\Delta)^K v - (\sigma-1)\,|u_k|^{\sigma-2}v\}$ with homogeneous Dirichlet boundary conditions, being

$$\Phi_0''(u_k)(v,v) \ =\ \big(((-\Delta)^K - (\sigma-1)\,|u_k|^{\sigma-2})v, v\big)_{K,2}$$

we have that

$$\sharp\{j \in \mathbb{N} :\ \mu_j \leqslant 0\} \ =\ m^*(\Phi_0, u_k) \ \geqslant\ k\,.$$

On the other hand, by applying Lemma 5.1 with

$$V(x) \ =\ \begin{cases} -(\sigma-1)\,|u_k(x)|^{\sigma-2} & \text{if} \quad x \in \Omega\,, \\ 0 & \text{if} \quad x \in \mathbb{R}^n \setminus \Omega\,, \end{cases}$$

there exists $B_{n,K} > 0$ such that

$$\sharp\{j \in \mathbb{N} :\ \mu_j \leqslant 0\} \ \leqslant\ B_{n,K}\, \big\||u_k|^{\sigma-2}\big\|_{n/2K}^{n/2K}\,.$$

It follows that

$$\|u_k\|_{n(\sigma-2)/2K}^{n(\sigma-2)/2K} \ \geqslant\ \beta k \tag{5.2}$$

for some $\beta > 0$. Moreover, $\big(\Phi_0'(u_k), u_k\big)_{K,2} = 0$ implies

$$b_k \ \geqslant\ \Phi_0(u_k) \ =\ \frac{\sigma-2}{2\sigma}\,\|u_k\|_\sigma^\sigma\,. \tag{5.3}$$

Since $\sigma > \frac{n(\sigma-2)}{2K}$, by (5.2) and (5.3) we have

$$b_k \ \geqslant\ C\,\|u_k\|_{n(\sigma-2)/2K}^\sigma \ \geqslant\ \alpha k^{2K\sigma/n(\sigma-2)}$$

for some $C, \alpha > 0$, which is the assertion. □

6 Proof of the results

By Lemmas 4.1, 4.2 and 4.3 the hypotheses $(\mathscr{B}_1)$, $(\mathscr{B}_2)$ and $(\mathscr{B}_3)$ are fulfilled. Moreover, for any finite dimensional subspace W of $H_0^K(\Omega)$ one has

$$\text{for all } u \in W: \quad \Phi_\theta(u) \leqslant \beta_1 \|u\|_{K,2}^2 - \beta_2 \|u\|_{K,2}^\sigma - \beta_3$$

for some constants $\beta_1, \beta_2, \beta_3 > 0$. Then,

$$\lim_{\substack{u \in W \\ \|u\|_{K,2} \to +\infty}} \sup_{\theta \in [0,1]} \Phi_\theta(u) = -\infty$$

and also $(\mathscr{B}_4)$ is satisfied.

Let now $f_1, \ldots, f_k$ be the first k eigenfunctions of $(-\Delta)^K$ with homogeneous boundary conditions, set $X_k = \mathbb{R} f_1 \oplus \ldots \oplus \mathbb{R} f_k$ and define the sequence (c_k) as in (2.1). We want to apply Theorem 2.1 by choosing η_1 and η_2 according to (4.6) of Lemma 4.3. Assume by contradiction that there exists $B > 0$ such that

$$\frac{|c_{k+1} - c_k|}{c_k^{\frac{\sigma-1}{\sigma}} + c_{k+1}^{\frac{\sigma-1}{\sigma}} + 1} \leqslant B\,. \tag{6.1}$$

In view of [1, Lemma 5.3], this yields $c_k \leqslant \gamma k^\sigma$ for some positive constant γ. Therefore, by Lemma 5.2 we conclude that (6.1) cannot hold provided that

$$\frac{2K\sigma}{n(\sigma - 2)} > \sigma\,, \tag{6.2}$$

namely $\sigma < 2\,\frac{n+K}{n}$, which concludes the proof of Theorem 1.1. □

Let us now assume $\phi_j = 0$ for each $j = 1, \ldots, K-1$. Arguing as above, by (4.7) of Lemma 4.3 one finds $\gamma' > 0$ such that $c_k \leqslant \gamma' k^{\sigma/(\sigma-1)}$. Therefore (6.2) becomes

$$\frac{2K\sigma}{n(\sigma - 2)} > \frac{\sigma}{\sigma - 1}\,,$$

which implies Theorem 1.2. □

Note that for each $n > 2K$ it results

$$2\,\frac{n+K}{n} < \frac{2n}{n-K} < K_*\,.$$

If $K = 1$, it has been proved in [5, 10] that Theorem 1.1 holds if

$$2 < \sigma < \frac{2n}{n-1}\,,$$

provided that Ω is of class C^2, $\phi_0 \in C^2(\partial\Omega)$ and $\varphi \in C(\overline{\Omega})$.

Conjecture 6.1 Let Ω be of class C^{2K}. Theorem 1.1 holds provided that

$$2 < \sigma < \frac{2n}{n-K}\,, \quad \phi_j \in C^{2K-j}(\partial\Omega) \quad j = 0, \ldots, K-1\,, \quad \varphi \in C(\overline{\Omega})\,.$$

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