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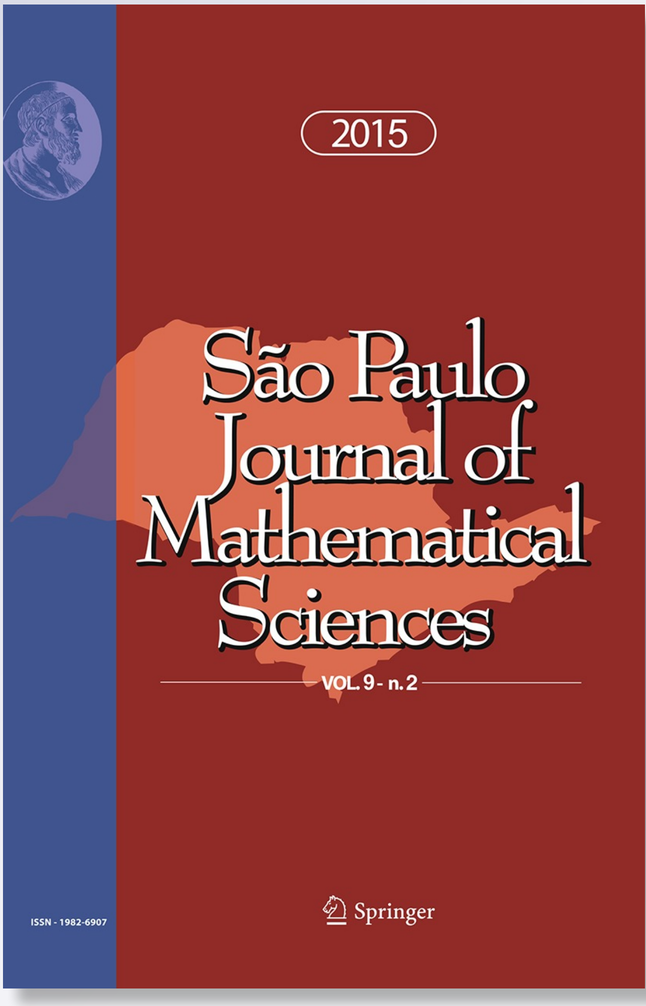
São Paulo Journal of Mathematical Sciences

ISSN 1982-6907
Volume 9
Number 2

São Paulo J. Math. Sci. (2015) 9:311-324
DOI 10.1007/s40863-015-0023-3



Springer

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DOI 10.1007/s40863-015-0023-3

Existence results for a doubly nonlocal equation

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Published online: 16 October 2015

Abstract In this note we expose some results proved in d'Avenia et al. [8] concerning an elliptic problem in $\mathbb{R}^N$ which involves two nonlocal operators: the fractional Laplacian and a convolution term of Hartree type. This equation has been called *fractional Choquard equation*. The results obtained concern regularity of weak solutions, existence and properties of ground states, as well as multiplicity and nonexistence of solutions.

Keywords Fractional Laplacian · Choquard equation · Existence · Nonexistence · Multiplicity

Dedicated to Prof. Orlando Lopes in occasion of his 70th birthday.

The first author was supported by GNAMPA project "*Aspetti differenziali e geometrici nello studio di problemi ellittici quasilineari*". The second author is supported by CNPq and FAPESP, Brazil. The third author was supported by GNAMPA project "*Problemi al contorno per operatori non locali non lineari*".

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Mathematics Subject Classification 35P15 · 35P30 · 35R11

1 Introduction

The equation we deal with is the following one:

$$(-\Delta)^s u + \omega u = \big(\mathcal{K}_\alpha * |u|^p\big)|u|^{p-2}u, \quad u \in H^s(\mathbb{R}^N),\ N \geq 3, \qquad (\mathcal{P}_\omega)$$

where $s \in (0, 1)$, $\omega > 0$ is a given parameter, $\alpha \in (0, N)$, $\mathcal{K}_\alpha(x) = |x|^{\alpha - N}$ is a convolution kernel and $p > 1$ belongs to a suitable interval to be specified later. The Hilbert space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \big\{u \in L^2(\mathbb{R}^N) : (-\Delta)^{s/2}u \in L^2(\mathbb{R}^N)\big\}$$

with its natural scalar product and norm given by

$$(u, v) = \int (-\Delta)^{s/2}u(-\Delta)^{s/2}v + \omega \int uv, \qquad \|u\|^2 = \|(-\Delta)^{s/2}u\|_2^2 + \omega\|u\|_2^2.$$

The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = -\frac{C(N, s)}{2}\int \frac{u(x + y) - u(x - y) - 2u(x)}{|y|^{N+2s}}dy, \quad x \in \mathbb{R}^N,$$

where $C(N, s)$ is a suitable normalization constant. As anticipated, problem $(\mathcal{P}_\omega)$ has nonlocal characteristics in the nonlinearity as well as in the (fractional) diffusion. When $s = 1$, $p = 2$ and $\alpha = 2$, then $(\mathcal{P}_\omega)$ reduces to the so-called Choquard or nonlinear Schrödinger–Newton equation

$$-\Delta u + \omega u = \Big(\mathcal{K}_2 * u^2\Big)u, \quad u \in H^1(\mathbb{R}^N)$$

on which there is a huge literature and appears in many phenomena: from quantum mechanics to self-gravitating matter theory; the interested reader is referred to the papers [4,14,16,20,21]. We have to say, that in these years the interests in the fractional Laplacian, and in general in pseudodifferential operators, has steadily grown: e.g. for $s = 1/2$, problem $(\mathcal{P}_\omega)$ has been used to model the dynamics of pseudo-relativistic boson stars, see [10]. The fractional Laplacian appears in the fractional Schrödinger equation by Laskin [12,13]; in [17,18] recent developments in the description of anomalous diffusion via fractional dynamics are discussed and fractional equations are derived asymptotically from Lévy random walk models, extending in a natural way Brownian walk models.

We will refer to $(\mathcal{P}_\omega)$ as to the generalized nonlinear Choquard equation.

Our aim is to give the main results on the existence and qualitative properties of weak solutions to $(\mathcal{P}_\omega)$. By a weak solution of $(\mathcal{P}_\omega)$ we mean a function $u \in H^s(\mathbb{R}^N)$ satisfying

$$\int (-\Delta)^{s/2}u\,(-\Delta)^{s/2}v + \omega \int uv = \int \big(\mathcal{K}_\alpha * |u|^p\big)|u|^{p-2}uv, \quad \text{for all } v \in H^s(\mathbb{R}^N).$$

In order to be everything well defined, we need to restrict the range of p to

$$1 + \frac{\alpha}{N} < p < \frac{N+\alpha}{N-2s}. \tag{1.1}$$

Therefore, without otherwise specified, this assumption will be everywhere tacitly assumed. We shall see however, that this condition turns out to be also necessary in order to get nontrivial solutions (i.e. $u \not\equiv 0$).

Solutions are found by variational methods; indeed they can be seen as critical points of the C^1 functional $E_\omega : H^s(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$E_\omega(u) = \frac{1}{2}\int |(-\Delta)^{s/2}u|^2 + \frac{\omega}{2}\int u^2 - \frac{1}{2p}\int \big(\mathcal{K}_\alpha * |u|^p\big)|u|^p.$$

In this context it is useful to introduce the Nehari manifold (which is a differentiable manifold of codimension one, bounded away from zero), which is formally obtained by multiplying the equation by u and integrating,

$$\mathcal{N}_\omega := \left\{u \in H^s(\mathbb{R}^N)\setminus\{0\} : \|(-\Delta)^{s/2}u\|_2^2 + \omega\|u\|_2^2 - \int \big(\mathcal{K}_\alpha * |u|^p\big)|u|^p = 0\right\}.$$

We say that a solution u of $(\mathcal{P}_\omega)$ is a *ground state* if its energy E_ω is minimal among all the solutions; hence, since all the solutions belong to $\mathcal{N}_\omega$, it can be characterized as $E(u) = \min_{v\in\mathcal{N}_\omega} E_\omega(v)$, if this minimum exists.

In the following we not give all the details of the proofs; we just present few simple proofs, and for the more involved one we simply give some ideas skipping the more technical and boring, and giving the precise reference of [8]. In Sect. 2 we present the main results and some ideas on how to address the proofs.

2 Main results and ideas of proofs

This section is divided into six subsection in which we address various aspects of the solutions of $(\mathcal{P}_\omega)$.

2.1 Regularity

The first result we present concerns the regularity of any weak solution.

Theorem 2.1 *Let u be a weak solution of $(\mathcal{P}_\omega)$. Then $u \in L^1(\mathbb{R}^N)$ and moreover*

- *if $s \le 1/2$, then $u \in C^{0,\mu}(\mathbb{R}^N)$ for $\mu \in (0, 2s)$;*
- *f $s > 1/2$, then $u \in C^{1,\mu}(\mathbb{R}^N)$ for $\mu \in (0, 2s-1)$.*

The proof is based on a standard and straightforward bootstrap argument. Without going into details, one first need to introduce the fractional Sobolev spaces:

$$\mathcal{W}^{\beta,q} = \{u \in L^q(\mathbb{R}^N) | \mathcal{F}^{-1}[(1+|\xi|^\beta)\mathcal{F}u] \in L^q(\mathbb{R}^N)\} \quad q \geq 1, \ \beta \geq 0,$$

see e.g. [23]. Then one show that if $u \in H^s(\mathbb{R}^N)$ is a weak solution of $(\mathcal{P}_\omega)$ and $r > 1$ then $u \in \mathcal{W}^{2s,r}$. This is done in [8, Lemma 3.3 and 3.4] by using some properties of the Bessel operator and the convolution integral. The regularity result is then obtained by combining this fact and the continuous embedding given in (ii) of the following

Proposition 2.2 (Theorem 3.2 of [9]) *We have:*

(i) If $\beta \geq 0$ and either $1 < r \leq q \leq r^*_\beta := Nr/(N-\beta r) < +\infty$ or $r = 1$ and $1 \leq q < N/(N-\beta)$, we have that $\mathcal{W}^{\beta,r}$ is continuously embedded in $L^q(\mathbb{R}^N)$.

(ii) Assume that $0 \leq \beta \leq 2$ and $\beta > N/r$. If $\beta - N/r < 1$ and $0 < \mu \leq \beta - N/r$ then $\mathcal{W}^{\beta,r}$ is continuously embedded in $C^{0,\mu}(\mathbb{R}^N)$. If $\beta - N/r > 1$ and $0 < \mu \leq \beta - N/r - 1$ then $\mathcal{W}^{\beta,r}$ is continuously embedded in $C^{1,\mu}(\mathbb{R}^N)$.

Beside the Hölder regularity just stated, it is worth noticing the next summability property of the fixed sign solutions. We will use this result in studying the Morse index. In this framework we need the energy functional E_ω to be C^2 and this is achieved, with a straightforward proof, for $p \geq 2$.

Proposition 2.3 *Let $s \in (1/2, 1)$ and $p \in [2, (N+\alpha)/(N-2s))$. If $u \in H^s(\mathbb{R}^N)$ is a solution of $(\mathcal{P}_\omega)$ with $|u| > 0$, then $u \in H^{2s+1}(\mathbb{R}^N)$. In particular $\nabla u \in H^s(\mathbb{R}^N)$.*

Remark 2.4 Under the hypotheses of the Proposition 2.3, we have $u \in C^2(\mathbb{R}^N)$. Indeed, since $u \in C^{1,\mu}(\mathbb{R}^N)$ with $\partial_i(-\omega u + (\mathcal{K}_\alpha * u^p)u^{p-1}) \in L^\infty(\mathbb{R}^N)$ and $\partial_i u$ satisfies

$$(-\Delta)^s \partial_i u = \partial_i \left(-\omega u + \left(\mathcal{K}_\alpha * u^p \right) u^{p-1} \right),$$

by [22, Proposition2.1.11], we conclude that $\partial_i u \in C^1(\mathbb{R}^N)$.

Coming back to the proof of Proposition 2.3, one has to show that $\|(-\Delta)^{s+1/2}u\|_2 < \infty$. For this, cut-off functions to deal with the convolution term inside and outside a ball in $\mathbb{R}^N$ are introduced. Finally usual properties of the convolution permits to conclude (see [8, Proposition 3.5]).

2.2 Asymptotics

Whenever $p \geq 2$ something more can be said on the solutions: for the sake of simplicity we set here

$$V := -\left(\mathcal{K}_\alpha * |u|^p \right) |u|^{p-2}.$$

The key observation now is that if a function u is in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, then $\mathcal{K}_\alpha * |u|^p \in C_0(\mathbb{R}^N)$, see [8, Lemma3.6]. We deduce that $V \in L^\infty(\mathbb{R}^N)$ and $V(x) \to 0$ for $|x| \to \infty$. Then an easy application of [11, LemmaC.2] allows us to obtain the asymptotic profile of the solutions as given in the next

Theorem 2.5 *Let $p \in [2, (N+\alpha)/(N-2s))$ and u be a solution of $(\mathcal{P}_\omega)$. Then there exist two positive constants C_1, C_2 such that, for any $x \in \mathbb{R}^N$,*

$$|u(x)| \leq C_1 \langle x \rangle^{-N-2s}, \quad \text{where } \langle x \rangle = (1+|x|^2)^{1/2}$$

and

$$u(x) = -C_2 \left(\int Vu \right) \frac{1}{|x|^{N+2s}} + o(|x|^{-N-2s}) \quad \text{for } |x| \to +\infty.$$

The restriction here on p is due to the fact we need a positive power of $|u|$ in V. Note that contrary to the local case $s = 1$, the solutions decay at a polynomial rate. We refer the reader to [19] for sharp results about the exponential decay of ground state solutions in the case $s = 1$.

2.3 Existence and further properties of the ground state

Once we have obtained the qualitative properties of (all) the solutions, we establish the existence results.

Ground states solutions have often a special interest. They are important both for a physical and mathematical point of view since they share further properties, like stability, positivity and symmetry. For $(\mathcal{P}_\omega)$ they can be found in various equivalent ways (see [8, Section 4]):

- by minimizing E_ω on $\mathcal{N}_\omega$,
- by minimizing E_0 (we mean E_ω with $\omega = 0$) on the sphere $\Sigma_\rho = \{u \in H^s(\mathbb{R}^N) : \|u\|_2 = \rho\}$ with $\rho > 0$,
- by minimizing

$$S(u) := \frac{\|u\|^2}{\left(\int (\mathcal{K}_\alpha * |u|^p)\, |u|^p\right)^{1/p}}$$

or

$$W(u) := \frac{\|(-\Delta)^{s/2} u\|_2^{\frac{N(p-1)-\alpha}{sp}} \, (\omega \|u\|_2^2)^{\frac{N+\alpha-(N-2s)p}{2sp}}}{\left(\int (\mathcal{K}_\alpha * |u|^p)\, |u|^p\right)^{1/p}}.$$

on $(H^s(\mathbb{R}^N) \cap L^{2Np/(N+\alpha)}(\mathbb{R}^N)) \setminus \{0\}$.

Arguing as in [19, Proof of Proposition 2.2] and applying [8, Lemma 2.2] we obtain

Theorem 2.6 *The functional S achieves the minimum on $H^s(\mathbb{R}^N) \setminus \{0\}$.*

The advantage of minimizing S instead of, e.g., E_0 on the constraint Σ_ρ, is that in this last case we need a further restriction on p: indeed to be well defined the minimization problem $\min_{\Sigma_\rho} E_0$ we have to assume $p \in (1+\alpha/N, 1+(2s+\alpha)/N)$ otherwise $\inf_{\Sigma_\rho} E_0 = -\infty$, as it is easily seen on the curve $\tau \mapsto \tau^{N/2} w(\tau \cdot)$, for a fixed $w \in \Sigma_\rho$.

However in the restricted range $p \in (1 + \alpha/N, 1 + (2s + \alpha)/N)$ it is possible to show by concentration compactness arguments that every minimizing sequence for E_0 on Σ_ρ is relatively compact. This fact is important in studying the orbital stability. As a byproduct, E_0 achieves the infimum on Σ_ρ, for every $\rho > 0$, and the minimum is a non-negative, radially symmetric and decreasing function, see [8, Theorem4.5].

The ground state has also a symmetry property as given in the next

Theorem 2.7 *Let $u \in H^s(\mathbb{R}^N)$ be a ground state of $(\mathcal{P}_\omega)$. Then u has fixed sign and there exist $x_0 \in \mathbb{R}^N$ and a monotone function $v : \mathbb{R} \to \mathbb{R}$ with fixed sign such that $u(x) = v(|x - x_0|)$.*

Proof Given a ground state u of $(\mathcal{P}_\omega)$, $u \neq 0$ and u is a solution of

$$S(u) = \inf_{\varphi \in H^s(\mathbb{R}^N) \setminus \{0\}} S(\varphi).$$

Taking into account $\|(-\Delta)^{s/2}|u|\|_2 \leq \|(-\Delta)^{s/2}u\|_2$ we see that also $|u|$ is a ground state and then satisfies

$$(-\Delta)^s|u| + \omega|u| = \big(\mathcal{K}_\alpha * |u|^p\big)|u|^{p-1}.$$

By arguing as in [9, end of Section 3], we see that $|u| > 0$ and u does not change sign: so we can assume $u > 0$. Given $v \in H^s(\mathbb{R}^N)$ with $v \geq 0$ and any half-space $H \subset \mathbb{R}^N$, the polarization v^H is defined as

$$v^H(x) = \begin{cases} \max\{v(x), v(\sigma_H(x))\} & \text{if } x \in H, \\ \min\{v(x), v(\sigma_H(x))\} & \text{if } x \in \mathbb{R}^N \setminus H, \end{cases}$$

where $\sigma^H(x)$ is the reflected of x with respect to ∂H. Then $\|v^H\|_2^2 = \|v\|_2^2$ and by [1, Theorem 2] we have $\|(-\Delta)^{s/2}v^H\|_2^2 \leq \|(-\Delta)^{s/2}v\|_2^2$. In turn, since $S(u) \leq S(u^H)$, we conclude that

$$\int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^p = \frac{\|u\|^{2p}}{[S(u)]^p} \geq \frac{\|u^H\|^{2p}}{[S(u^H)]^p} = \int \left(\mathcal{K}_\alpha * |u^H|^p\right)|u^H|^p.$$

Then, by combining [19, Lemma 5.3 and Lemma5.4], we conclude the proof. □

Finally we study the Morse index of the ground state. Here the details are given. We assume $2 \leq p < 1 + (2s + \alpha)/N$, to have the functional E_ω of class C^2, and $s > 1/2$. If u is the minimum of E_0 on Σ_ρ (recall this is an equivalent characterization of the ground state) we have

$$\int |(-\Delta)^{s/2}u|^2 - \int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^p = -\lambda\rho^2 \tag{2.1}$$

and easy computations show that $\lambda > 0$. Now consider

$$E''_\lambda(u)[\xi,\eta] = \int (-\Delta)^{s/2}\xi(-\Delta)^{s/2}\eta + \lambda\int \xi\eta \\ -p\int \left(\mathcal{K}_\alpha * |u|^{p-2}u\eta\right)|u|^{p-2}u\xi - (p-1)\int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^{p-2}\xi\eta \tag{2.2}$$

and let's go to study its kernel. Since the problem is invariant for the group of translations, the solutions of $(\mathcal{P}_\omega)$ will never be isolated, then $\ker E''_\lambda(u) \neq \{0\}$ and in fact we will prove that

$$\operatorname{span}\{\nabla u\} \subset \ker E''_\lambda(u). \tag{2.3}$$

Indeed, for every $a \in \mathbb{R}^N$, consider the linear and isometric action of the group of the translations in $\mathbb{R}^N$ induced on $H^s(\mathbb{R}^N)$, that is

$$\mathfrak{t}_a : u \in H^s(\mathbb{R}^N) \longmapsto u(\cdot + a) \in H^s(\mathbb{R}^N).$$

Since $E_\lambda \circ \mathfrak{t}_a = E_\lambda$, we have $E'_\lambda(\mathfrak{t}_a u)[v] = E'_\lambda(u)[\mathfrak{t}_{-a}v]$, for every $u, v \in H^s(\mathbb{R}^N)$. For every $u \in H^s(\mathbb{R}^N)$ it is also convenient to introduce the following map

$$\mathfrak{s}_u : a \in \mathbb{R}^N \longmapsto u(\cdot + a) \in H^s(\mathbb{R}^N).$$

Of course, for a generic fixed $u \in H^s(\mathbb{R}^N)$, the map $\mathfrak{s}_u$ does not need to be differentiable but (for example) whenever $u \in H^s(\mathbb{R}^N)$ is a solution of $(\mathcal{P}_\omega)$ as in Proposition 2.3 it does, and the differential in 0 is given by

$$\mathfrak{s}'_u(0)[b] = \nabla u \cdot b \in H^s(\mathbb{R}^N), \qquad \text{for all } b \in \mathbb{R}^N.$$

Hence, in this case, by differentiating in 0 the map

$$a \in \mathbb{R}^N \longmapsto E'_\lambda(\mathfrak{s}_u(a)) \in H^{-s}(\mathbb{R}^N),$$

we get $E''_\lambda(\mathfrak{s}_u(0))[\mathfrak{s}'_u(0)[b], \cdot] = 0$ for all $b \in \mathbb{R}^N$ and this gives (2.3).

It would be interesting to understand if the ground state is nondegenerate in the sense that

$$\operatorname{span}\{\nabla u\} = \ker E''_\lambda(u).$$

The Morse index $\mathrm{i}_{\mathrm{Morse}}(u)$ is defined as the maximal dimension of subspaces of $H^s(\mathbb{R}^N)$ on which $E''_\lambda(u)$ is negative definite.

Proposition 2.8 *Let $u \in \Sigma_\rho$ be a ground state and $T_u\Sigma_\rho = \{w \in H^s(\mathbb{R}^N) : \int uw = 0\}$. Then*

(i) $E''_\lambda(u)$ *is positive semidefinite on* $T_u\Sigma_\rho$,
(ii) $\inf_{w\in T_u\Sigma_\rho} E''_\lambda(u)[w,w] = 0$.
(iii) $\mathrm{i}_{\mathrm{Morse}}(u) = 1$.

Proof Let v any element of $T_u\Sigma_\rho$ and $\gamma : (-\varepsilon, \varepsilon) \to \Sigma_\rho$ a smooth curve such that $\gamma(0) = u$ and $\gamma'(0) = v$. Since u is the minimum of E_0 on Σ_ρ, it is

$$\frac{d^2}{d\tau^2} E_0(\gamma(\tau))\Big|_{\tau=0} \geq 0$$

which explicitly reads as

$$0 \leq E_0''(u)[v, v] + E_0'(u)[\gamma''(0)] = E_0''(u)[v, v] - \lambda \int u\gamma''(0). \tag{2.4}$$

Of course, $0 = \frac{d}{d\tau} \int |\gamma(\tau)|^2 = 2 \int \gamma(\tau)\gamma'(\tau)$ implies

$$\int |v|^2 + \int u\gamma''(0) = 0,$$

which, plugged into (2.4) gives (i). Property (ii) follows by Proposition 2.3 and the translation invariance of Σ_ρ: indeed $\partial_{x_i} u \in T_u\Sigma_\rho$ and we know $E_\lambda''(u)[\partial_{x_i} u, \partial_{x_i} u] = 0$. Finally, to prove (iii), note that by (2.2) and (2.1)

$$\begin{aligned} E_\lambda''(u)[u, u] &= \int |(-\Delta)^{s/2} u|^2 + \lambda\rho^2 + (1 - 2p) \int \left(\mathcal{K}_\alpha * |u|^p\right) |u|^p \\ &= 2(1 - p) \int \left(\mathcal{K}_\alpha * |u|^p\right) |u|^p < 0. \end{aligned}$$

The result then follows from (i) and the direct sum decomposition (see [2] for the general setting): $H^s(\mathbb{R}^N) = T_u\Sigma_\rho \oplus \text{span}\{u\}$. □

2.4 Multiplicity of bound states

The problem under consideration has also other type of solutions (actually a sequence) which in general are changing sign. These solutions are of "mountain pass" type, in the sense they are obtained by minimax arguments. To apply these methods, the functional E_ω has to satisfy some geometric and compactness properties. The geometric properties are listed in the next

Proposition 2.9 *The functional E_ω satisfies the following geometric assumptions of the Symmetric Mountain Pass Theorem:*

(i) it is even, that is $E_\omega(u) = E_\omega(-u)$,
(ii) it has has a strict local minimum in 0 with $E_\omega(0) = 0$,
(iii) there exist a nested sequence $\{V_k\}$ of finite dimensional subspaces of $H^s(\mathbb{R}^N)$ and $\{R_k\} \subset \mathbb{R}^+$ such that $E_\omega(u) \leq 0$ for every $u \in V_k$ with $\|u\| \geq R_k$.

Proof Property (i) is immediate. By standard inequality it holds

$$E_\omega(u) \geq \frac{1}{2}\|u\|^2 - C\|u\|^{2p}$$

getting (ii). Finally, if $\{e_i\}_{i=1,\dots,k}$ is an orthogonal basis of a $k-$dimensional subspace V_k of $H^s(\mathbb{R}^N)$, then, writing $u = \sum_{i=1}^k t_i e_i$, it is $E_\omega(u) \to -\infty$ for $\|u\| \to \infty$, proving (iii). □

For what concerns compactness we need to restrict to functions with some symmetries. To this aim some preliminaries are in order.

Firstly, let $\ell > 1$, $N_i \geq 2$, $i = 1,\dots,\ell$, or $\ell = 1$ and $N \geq 3$, and $N = \sum_{i=1}^{\ell} N_i$. A point in $\mathbb{R}^N$ is now denoted with $x = (x_1,\dots,x_\ell)$, $x_i \in \mathbb{R}^{N_i}$. Let $\mathscr{O}(N_i)$ be the orthogonal group on $\mathbb{R}^{N_i}$ and consider the product group

$$G := \mathscr{O}(N_1) \times \cdots \times \mathscr{O}(N_\ell)$$

acting on $\mathbb{R}^N$ by

$$g \cdot x = (g_1 x_1, \dots, g_\ell x_\ell), \quad g = (g_1,\dots,g_\ell) \in G$$

and whose representation in $H^s(\mathbb{R}^N)$ is given by the linear and isometric action

$$(T_g u)(x) = u(g^{-1} \cdot x). \tag{2.5}$$

Set

$$X := \{u \in H^s(\mathbb{R}^N) : T_g u = u \text{ for all } g \in G\}.$$

In particular for $\ell = 1$ we have the radial functions, $u(x) = u(|x|)$. We say that the functions in X are "symmetric". Then X is exactly the closed and infinite dimensional subspace of fixed points for the action (2.5). The importance of this setting is twofold. Indeed the functional E_ω is $G-$invariant, i.e. for every $g \in G$, $E_\omega \circ T_g = E_\omega$ and the space X has compact embedding into $L^q(\mathbb{R}^N)$, $q \in (2, 2_s^*)$, see [15].

Secondly, for every fixed $u \in H^s(\mathbb{R}^N)$, consider the problem

$$\begin{cases} (-\Delta)^{\alpha/2}\varphi = \gamma(\alpha)|u|^p, \quad \text{where } \gamma(\alpha) := \frac{\pi^{N/2}2^\alpha\Gamma(\alpha/2)}{\Gamma(N/2-\alpha/2)}, \\ \varphi \in \dot{H}^{\alpha/2}(\mathbb{R}^N), \end{cases} \tag{2.6}$$

(where Γ is the gamma function) whose weak formulation is the following one: we say that $\varphi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$ is a weak solution if for every $\xi \in \dot{H}^{\alpha/2}(\mathbb{R}^N)$

$$\int (-\Delta)^{\alpha/4}\varphi(-\Delta)^{\alpha/4}\xi = \gamma(\alpha)\int \xi|u|^p. \tag{2.7}$$

Recall that for every $\alpha \in (0, N)$, $(-\Delta)^{\alpha/2}u$ is defined via the Fourier transform and $\dot{H}^{\alpha/2}(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the associated Gagliardo seminorm $\|(-\Delta)^{\alpha/4}u\|_2$ (these notions coincide with that given in the Introduction for $\alpha \in (0, 2)$). Observe now that, under the assumption on p, the right hand side in (2.7) defines the map

$$L : v \in \dot{H}^{\alpha/2}(\mathbb{R}^N) \mapsto \int v|u|^p \in \mathbb{R}$$

which is linear and continuous; indeed

$$|Lv| \leq C\|u\|^p_{2Np/(N+\alpha)}\|v\|_{\dot{H}^{\alpha/2}} \leq C\|u\|^p\|v\|_{\dot{H}^{\alpha/2}}.$$

By the Riesz Representation Theorem there exists a unique weak solution φ of (2.6), represented as a convolution with the kernel $\mathcal{K}_\alpha/\gamma(\alpha)$, i.e. $\varphi = \mathcal{K}_\alpha * |u|^p$ (see e.g. [23]) and

$$\|\mathcal{K}_\alpha * |u|^p\|_{\dot{H}^{\alpha/2}} = \|L\| \leq C\|u\|^p.$$

As a consequence of the above setting we can prove the following result, which will help us to recover compactness.

Lemma 2.10 *Let $\{u_n\}, u \in X$ be such that $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. Then*

(i) $\mathcal{K}_\alpha * |u_n|^p \to \mathcal{K}_\alpha * |u|^p$ *in* $\dot{H}^{\alpha/2}(\mathbb{R}^N)$*;*
(ii) $\int (\mathcal{K}_\alpha * |u_n|^p)\,|u_n|^p \to \int (\mathcal{K}_\alpha * |u|^p)\,|u|^p$*;*
(iii) $\int (\mathcal{K}_\alpha * |u_n|^p)\,|u_n|^{p-2}u_n u \to \int (\mathcal{K}_\alpha * |u|^p)\,|u|^p$*.*

We omit the easy proof which uses standard argument as Young inequality and Dominated Convergence Theorem [8, Lemma 5.2].

Proposition 2.11 *The functional E_ω satisfies the Palais–Smale condition in X.*

Proof Let $\{u_n\} \subset X$ be a Palais–Smale sequence, that is,

$$|E_\omega(u_n)| \leq M, \quad E'_\omega(u_n) \to 0 \ \text{ in } \ H^{-s}(\mathbb{R}^N).$$

Then we deduce in a standard way the boundedness of $\{u_n\}$ in $H^s(\mathbb{R}^N)$. Hence, there exists $u \in X$ such that, up to subsequences, $u_n \rightharpoonup u$ in $H^s(\mathbb{R}^N)$. By Lemma 2.10 we have the convergences

$$\begin{aligned} 0 \longleftarrow E'_\omega(u_n)[u] &= (u_n, u) - \int \left(\mathcal{K}_\alpha * |u_n|^p\right)|u_n|^{p-2}u_n u \longrightarrow \|u\|^2 \\ &\quad - \int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^p, \\ E'_\omega(u_n)[u_n] &= \|u_n\|^2 - \int \left(\mathcal{K}_\alpha * |u_n|^p\right)|u_n|^p \longrightarrow 0, \\ \int \left(\mathcal{K}_\alpha * |u_n|^p\right)|u_n|^p &\longrightarrow \int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^p, \end{aligned}$$

from which we deduce that $\|u_n\| \to \|u\|$, concluding the proof. □

We can prove now the following

Theorem 2.12 *The functional E_ω possesses infinitely many critical points $\{u_n\} \subset X$ such that $E_\omega(u_n) \to \infty$, and $\|u_n\| \to \infty$. In paticular, problem $(\mathcal{P}_\omega)$ has infinitely many solutions in X.*

Proof All the hypotheses (geometry and compactness) of the Symmetric Mountain Pass Theorem on the space X are satisfied, so that the existence of infinitely many critical points $\{u_n\} \subset X$ with $E_\omega(u_n) \to \infty$ is guaranteed. Then, since $\int (\mathcal{K}_\alpha * |u_n|)|u_n|^p \le C\|u_n\|^{2p}$, it has to be $\|u_n\| \to \infty$. By the Palais Principle of Symmetric Criticality, the constrained critical points $\{u_n\} \subset X$ for E_ω are indeed "true" critical points and hence solutions of $(\mathcal{P}_\omega)$. □

Observe that Proposition 2.9 holds also in the limit cases $p = 1 + \alpha/N$ and $p = (N + \alpha)/(N - 2s)$. Due to the nonexistence result (see Sect. 2.5), we see that the Palais–Smale condition cannot be satisfied for these values.

Note however that up to this point we may have found purely radial solutions, in the sense that we cannot distinguish between the cases $\ell = 1$ or $\ell > 1$. To obtain genuine nonradial solutions we need a slight modification in the above setting, as introduced in [3]. Unfortunately a restriction on the dimension N appears here. Let $N = 4$ or $N \ge 6$ and choose an integer $m \ne (N - 1)/2$ such that $2 \le m \le N/2$. Let us define

$$G := \mathscr{O}(m) \times \mathscr{O}(m) \times \mathscr{O}(N - 2m)$$

whose induced action on $H^s(\mathbb{R}^N)$ is as usual

$$(T_g u)(x) = u\left(g_1^{-1}x_1, g_2^{-1}x_2, g_3^{-1}x_3\right), \quad g = (g_1, g_2, g_3) \in G, \tag{2.8}$$

where, now $x = (x_1, x_2, x_3) \in \mathbb{R}^m \oplus \mathbb{R}^m \oplus \mathbb{R}^{N-2m}$. We know that X, associated to the action (2.8), has compact embedding into $L^q(\mathbb{R}^N)$, $q \in (2, 2_s^*)$. The key point now consists in considering the involution in $\mathbb{R}^N$

$$\tau \cdot x = (x_2, x_1, x_3)$$

and the action

$$(\mathcal{I}u)(x) = u(x), \quad (\mathcal{T}u)(x) = -u(\tau^{-1} \cdot x)$$

induced by $H = \{\iota_H, \tau\}$ on $H^s(\mathbb{R}^N)$. Define the semidirect product

$$K := G \rtimes_\psi H \subset \mathscr{O}(N)$$

via the group homomorphism $\psi : H \to \mathrm{Aut}(G)$ given by

$$\psi(\iota_H)g = g, \quad \psi(\tau)g = g^{-1}, \quad g \in G.$$

Moreover, if

$$\pi : K \to \{+1, -1\} \text{ such that } \pi(g, \iota_H) = 1, \ \ \pi(g, \tau) = -1$$

denotes the canonical epimorphism, we define the action of K on $H^s(\mathbb{R}^N)$ by

$$(T_k u)(x) := \pi(k)u(k^{-1} \cdot x), \ k \in K.$$

Of course, this action is linear and isometric and in particular if, $k = (g, \iota_H)$ then $(T_k u)(x) = u(g^{-1} \cdot x)$, if, $k = (\iota_G, \tau)$ then $(T_k u)(x) = -u(\tau^{-1} \cdot x)$. Set

$$Y := \{u \in H^s(\mathbb{R}^N) : T_k u = u \text{ for all } k \in K\}$$

and note that the unique radial function in Y is $u \equiv 0$. Since E_ω is $K-$invariant and $Y \subset X$ is closed and infinite dimensional, we can argue as before to obtain the following multiplicity result.

Theorem 2.13 *Assume $N = 4$ or $N \geq 6$. The functional E_ω possesses infinitely many critical points $\{u_n\} \subset Y$ such that $E_\omega(u_n) \to \infty$ and $\|u_n\| \to \infty$. In particular, problem $(\mathcal{P}_\omega)$ has infinitely many solutions in Y.*

2.5 Nonexistence result

This subsection justifies the range in which p varies.

Theorem 2.14 *Assume that either $p \leq 1 + \alpha/N$ or $p \geq (N + \alpha)/(N - 2s)$. Then $(\mathcal{P}_\omega)$ does not admit nontrivial solutions $u \in C^2(\mathbb{R}^N)$.*

As a consequence, the range of p detected in (1.1) is optimal for the existence of nontrivial solutions. Theorem 2.14 is based upon a Pohožaev identity

$$(N - 2s) \int |(-\Delta)^{s/2}u|^2 + \omega N \int |u|^2 = \frac{\alpha + N}{p} \int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^p. \tag{2.9}$$

The proof is technical and is obtained, as in [6], by the localization procedure due to Caffarelli and Silvestre [5].

The proof of the theorem is then achieved by combining the Pohožaev Identity with the fact that any solution satisfies

$$\int |(-\Delta)^{s/2}u|^2 + \omega \int |u|^2 = \int \left(\mathcal{K}_\alpha * |u|^p\right)|u|^p. \tag{2.10}$$

Indeed combining (2.9) and (2.10) we get

$$\left(N - 2s - \frac{\alpha + N}{p}\right) \int |(-\Delta)^{s/2}u|^2 + \omega \left(N - \frac{\alpha + N}{p}\right) \int |u|^2 = 0$$

and hence, since $\omega > 0$, if both the coefficients are positive, that is $p \geq \alpha + N/(N - 2s)$, the unique solution is the trivial one. Analogously, if they are both negative, that is $p \leq 1 + \alpha/N$, nontrivial solutions cannot exist.

2.6 The "zero mass" case

A second problem addressed in the paper [8] related to equation ($\mathcal{P}_\omega$) is the so called "zero mass" problem:

$$(-\Delta)^s u = \big(\mathcal{K}_\alpha * |u|^p\big)|u|^{p-2}u, \quad u \in \dot{H}^s(\mathbb{R}^N) \qquad (\mathcal{P}_0)$$

We have established the following result.

Theorem 2.15 *The following assertions hold:*

(1) Let $p \neq \frac{\alpha+N}{N-2s}$. Then $(\mathcal{P}_0)$ does not admit nontrivial solutions $u \in \dot{H}^s(\mathbb{R}^N) \cap L^{\frac{2pN}{N+\alpha}}(\mathbb{R}^N)$.

(2) Let $p = \frac{\alpha+N}{N-2s} = 2$. Then the problem writes as

$$(-\Delta)^s u = (|x|^{-4s} * |u|^2)u, \quad u \in \dot{H}^s(\mathbb{R}^N), \quad N > 4s, \tag{2.11}$$

and any of its solutions of fixed sign have the form

$$C\left(\frac{t}{t^2 + |x-x_0|^2}\right)^{\frac{N-2s}{2}}, \quad x \in \mathbb{R}^N, \tag{2.12}$$

for some $x_0 \in \mathbb{R}^N$, $C > 0$ and $t > 0$.

The classification of the solutions to problem (2.11) is reminiscent of that for the fixed-sign solutions to

$$(-\Delta)^s u = u^{\frac{N+2s}{N-2s}} \qquad \text{in } \mathbb{R}^N.$$

In fact in [7] the authors proved that any positive solution to this problem has the form of (2.12).

Finally the first statement of Theorem 2.15 follows by Pohožaev identity (2.9), while for the second assertion we refer the reader to the final part in [8] which is a little bit technical, involving the Kelvin transform, moving plane methods and asymptotic decay.

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