ON A CLASS ON NONLINEAR ELLIPTIC BVP AT CRITICAL GROWTH

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Abstract. Existence and multiplicity of solutions for a general class of quasilinear elliptic problems at critical growth with perturbations of lower order are considered. Techniques of nonsmooth critical point theory are employed.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < n$. This paper is devoted to the existence and multiplicity of solutions in $W^{1,p}_0(\Omega)$ to the following class of nonlinear elliptic BVP, say $\mathcal{C}_g$, involving the critical Sobolev exponent $p^* = \frac{np}{n-p}$

\[
\begin{cases}
- \text{div} \left( j_\xi(x, u, \nabla u) \right) + j_s(x, u, \nabla u) = |u|^{p^*-2}u + g(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

As a particular case, for $p < q < p^*$, we consider problems, say $\mathcal{C}_{\varepsilon,\lambda}$

\[
\begin{cases}
- \text{div} \left( j_\xi(x, u, \nabla u) \right) + j_s(x, u, \nabla u) = |u|^{p^*-2}u + \lambda |u|^{q-2}u + \varepsilon h & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

with $h \in L^q(\Omega)$, $h \neq 0$, $\lambda > 0$ and $\varepsilon > 0$.

Motivations for investigating $\mathcal{C}_g$ and $\mathcal{C}_{\varepsilon,\lambda}$ come from various situations in geometry and physics which involve lack of compactness (see e.g. [5]). Since, as known, the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails to be compact, one encounters serious difficulties in applying variational methods.

We refer the reader to [5] for the case $j = -\Delta$ and to [3, 9, 10] for the extension to degenerate operators ($j = -\Delta_p$, $p \neq 2$). For the existence of multiple solutions (two) for

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problems $\mathcal{C}_{g,\lambda}$, we refer the reader to [17] for $j = -\Delta$ and to [8] for $j = -\Delta_p$. In these cases the associated functional is smooth ($C^1$).

On the other hand, under natural assumptions on $j$, the functional associated to $\mathcal{C}_g$ and $\mathcal{C}_{g,\lambda}$ is merely continuous (not even locally Lipschitz continuous) unless $j$ does not depend on $u$ or it is subjected to some restrictive growth conditions. Therefore, in general one also has lack of regularity and techniques of nonsmooth critical point theory have to be employed (see e.g. [6, 7] and references therein).

Quite recently, some existence results for problem

$$
\begin{cases}
- \text{div} (j(x,u,\nabla u)) + j_s(x,u,\nabla u) = b(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

with $b$ subcritical and superlinear have been considered in [1, 12] and e.g. in [6, 7, 14] via different points of view. It is therefore natural to wonder what happens when $b$ reaches the critical growth and has some subcritical perturbation $g \neq 0$.

The first existence result in this framework was given in [2] for

$$j(x,s,\xi) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i \xi_j, \quad b(x,u) = \lambda u + |u|^{2s-2}u,$$

provided that the $a_{ij}$s satisfy some suitable assumptions, including an asymptotic behaviour as $s$ goes to $+\infty$ (cf. conditions (10) and (11)).

In view of this result, it is expected that under suitable assumptions on $g$ and $j$ problems $\mathcal{C}_g$ admit at least one nontrivial solution and problems $\mathcal{C}_{g,\lambda}$ admit at least two nontrivial solutions for $\lambda > 0$ large and $\varepsilon > 0$ small (which depends on $\lambda$). The goal of this paper is precisely to prove these results thus extending the achievements of [3, 5, 8, 9, 10, 17] to this general and unified setting.

To carry on our analysis, we look for critical points (in a suitable sense) of the functional $f : W^{1,p}_0(\Omega) \to \mathbb{R}$ given by

$$f(u) = \int_{\Omega} j(x,u,\nabla u) \, dx - \frac{1}{p^*} \int_{\Omega} |u|^{p^*} \, dx - \int_{\Omega} G(x,u) \, dx$$

(1)

where $G(x,s) = \int_{0}^{s} g(x,t) \, dt$ and $W^{1,p}_0(\Omega)$ is endowed with the standard norm $\|u\|_{1,p} = \left(\int_{\Omega} |\nabla u|^p \, dx\right)^{1/p}$.

We assume that $j(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is measurable in $x$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$, of class $C^1$ in $(s,\xi)$ and that $j(x,s,\cdot)$ is strictly convex and $p-$homogeneous with $j(x,s,0) = 0$. Moreover:

($\mathcal{A}_1$) there exist $\nu > 0$ and $c_1, c_2 > 0$ such that:

$$j(x,s,\xi) \geq \frac{\nu}{p} |\xi|^p, \quad |j_s(x,s,\xi)| \leq c_1 |\xi|^p,$$

(2)
a.e. in \( \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\) and

\[
|j_\xi(x, s, \xi)| \leq c_2|\xi|^{p-1},
\]

(3)
a.e. in \( \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\);

(\(\mathcal{A}_2\)) there exist \(R, R' > 0\) and \(\gamma \in (0, p^* - p)\) such that:

\[
|s| \geq R \implies j_s(x, s, \xi)s \geq 0,
\]

(4)
\[
|s| \geq R' \implies j_s(x, s, \xi)s \leq \gamma j(x, s, \xi),
\]

(5)
a.e. in \( \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\);

(\(\mathcal{A}_2\)) Let \(\lambda_1\) be the first eigenvalue of \(-\Delta_\xi\) with homogeneous boundary conditions and \(g : \Omega \times \mathbb{R} \to \mathbb{R}\) a Carathéodory map with \(g(x, 0) = 0\) and:

\[
\forall \varepsilon > 0 \ \exists a_\varepsilon \in L^{\frac{np}{n(p-1)+p}}(\Omega) : |g(x, s)| \leq a_\varepsilon(x) + \varepsilon|s|^{p^*-1},
\]

(6)
\[
\limsup_{s \to 0} \frac{G(x, s)}{|s|^p} < \frac{v\lambda_1}{p}, \quad G(x, s) \geq 0,
\]

(7)
uniformly in \( \Omega \). Moreover, we assume that there exists a nonempty open set \(\Omega_0 \subset \Omega\) such that:

- if \(n < p^2\) (critical dimensions):

\[
\lim_{s \to +\infty} \frac{G(x, s)}{s^{p(np+p-2n)/(p-1)(n-p)}} = +\infty,
\]

(8)
uniformly in \(\Omega_0\).

- if \(n \geq p^2\): \(\exists \mu > 0, \exists b > a:\)

\[
\forall s \in [a, b] : G(x, s) \geq \mu
\]

(9)
for a.e. \(x \in \Omega_0\) (\(\mu\) sufficiently large in the case \(n = p^2\)).

Conditions (2), (3), (4) and (5) have already been considered e.g. in \([1, 12, 14]\), while assumptions (6), (7), (8) and (9) can be found in \([3]\). Note that \(g(x, u)\) is neither assumed to be positive nor homogeneous in \(u\).

Assume now furthermore that (asymptotic behaviour):

\[
\lim_{s \to +\infty} j(x, s, \xi) = \frac{1}{p}|\xi|^p, \quad (10)
\]

\[
\lim_{s \to +\infty} j_s(x, s, \xi)s = 0, \quad (11)
\]

uniformly with respect to \(x \in \Omega\) and to \(\xi \in \mathbb{R}^n\) with \(|\xi| \leq 1\). This means that there exist \(\varepsilon_1 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) and \(\varepsilon_2 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}\) such that:

\[
j(x, s, \xi) = \frac{1}{p}|\xi|^p + \varepsilon_1(x, s, \xi)|\xi|^p, \quad j_s(x, s, \xi)s = \varepsilon_2(x, s, \xi)|\xi|^p
\]
where $\varepsilon_{1,2}(x, s, \xi) \to 0$ as $s \to +\infty$ uniformly for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Under assumptions $(A_1)$, $(A_2)$, $(A_3)$, (10) and (11), in Section 4 we will prove the following result.

**Theorem 1.1.** $C_g$ admits at least one nontrivial positive solution.

This result extends the achievements of [2, 3] to a more general class of elliptic boundary value problems and gives a more complete picture of the results of [16].

Assume now that $\gamma \in (0, q - p)$ and $R' = 0$ in (5) and (12) holds. Moreover, let $j(x, s, \cdot)$ be of class $C^2$ and $b_1 > 0$ with

$$|j_{\xi}(x, s, \xi)| \leq b_1|\xi|^{p-2}$$

(12)
a.e. in $\Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Under assumptions $(A_1)$, $(A_2)$ and (12), we have the following result.

**Theorem 1.2.** $C_{\varepsilon, \lambda}$ admits at least two nontrivial solutions provided that $\lambda > 0$ is sufficiently large and $\varepsilon > 0$ is sufficiently small (depending on $\lambda$).

For the full proof of this result, we refer the reader to [16]. In this paper, we prefer to prove in Section 5 a general version of the compactness theorem. This result extends the achievements of [8] to a more general class of elliptic boundary value problems. We stress that we proved our result without any use of concentration-compactness techniques [11]. From this point of view, our approach seems to be simpler and more direct.

Assume finally that $\Omega$ is star-shaped, $h = 0$, $\lambda \leq 0$ and

$$p^*j_x(x, s, \xi) \cdot x - n j_s(x, s, \xi)s \geq 0$$
a.e. in $\Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. Then from the general variational identity of Pucci–Serrin [13], we derive the following result.

**Theorem 1.3.** $C_{\varepsilon, \lambda}$ admits no nontrivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

For the proof, we refer the reader to [16, Corollary 6.2].

## 2 Recalls of nonsmooth analysis

Let $(X, d)$ be a metric space and let $f : X \to \mathbb{R}$ be a continuous function.

**Definition 2.1.** (see [7]). For every $u \in X$ we denote by $|df|(u)$ the supremum of $\sigma s$ in $[0, +\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathcal{H} : B_\delta(u) \times [0, \delta] \to X$ such that

$$\forall v \in B_\delta(u) \land \forall t \in [0, \delta] : d(\mathcal{H}(v, t), v) \leq t,$$

$$\forall v \in B_\delta(u) \land \forall t \in [0, \delta] : f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called the weak slope of $f$ at $u$. 

If \( f \) is of class \( C^1 \) and \( X \) is normed, \(|df|(u) = \|df(u)\|_X\) for each \( u \in X \).

**Definition 2.2.** We say that \( u \in X \) is a critical point for \( f \) if \(|df|(u) = 0 \). Let \( c \in \mathbb{R} \). We say that \((u_h)\) is a Palais–Smale sequence for \( f \) at level \( c \) \(((PS)_c \text{–sequence})\) if \( f(u_h) \to c \) and \(|df|(u_h) \to 0 \). We say that \( f \) satisfies the Palais–Smale condition at level \( c \) if every \((PS)_c \text{–sequence} \) for \( f \) admits a convergent subsequence.

**Definition 2.3.** A sequence \((u_h) \subset W^{1,p}_0(\Omega)\) is said to be a concrete Palais–Smale sequence at level \( c \in \mathbb{R} \) \(((CPS)_c \text{–sequence})\) for \( f \), if \( f(u_h) \to c \),

\[
-\text{div} \left( j_\xi(x, u_h, \nabla u_h) \right) + j_s(x, u_h, \nabla u_h) \in W^{-1,p'}(\Omega),
\]
eventually as \( h \to +\infty \) and

\[
-\text{div} \left( j_\xi(x, u_h, \nabla u_h) \right) + j_s(x, u_h, \nabla u_h) - |u_h|^{p^*-2} u_h - g(x, u_h) \to 0
\]

strongly in \( W^{-1,p'}(\Omega) \). We say that \( f \) satisfies the concrete Palais–Smale condition at level \( c \) \(((CPS)_c \text{ in short})\), if every \((CPS)_c \text{–sequence} \) for \( f \) admits a strongly convergent subsequence.

**Definition 2.4.** We say that \( u \) is a weak solution to \( \mathcal{C}_g \) if \( u \in W^{1,p}_0(\Omega) \) and

\[
-\text{div} \left( j_\xi(x, u, \nabla u) \right) + j_s(x, u, \nabla u) = |u|^{p^*-2} u + g(x, u)
\]
in distributional sense.

**Proposition 2.5.** Let \( u \in W^{1,p}_0(\Omega) \) be such that \(|df|(u) < +\infty \). Then

\[
w_u := -\text{div} \left( j_\xi(x, u, \nabla u) \right) + j_s(x, u, \nabla u) - |u|^{p^*-2} u - g(x, u)
\]
belongs to \( W^{-1,p'}(\Omega) \) and \( \|w_u\|_{-1,p'} \leq |df|(u) \).

In particular, if \( u \) is a critical point of \( f \) then \( u \) is a weak solution to \( \mathcal{C}_g \). Finally, it is readily seen by the above Proposition that if \( f \) satisfies \((CPS)_c\), then is satisfies \((PS)\).

### 3 Existence of one nontrivial solution

Let us set for a.e. \( x \in \Omega \) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\)

\[
\tilde{j}(x, s, \xi) = \begin{cases} j(x, s, \xi) & \text{if } s \geq 0 \\ j(x, 0, \xi) & \text{if } s < 0, \end{cases} \quad \tilde{g}(x, s) = \begin{cases} g(x, s) & \text{if } s \geq 0 \\ 0 & \text{if } s < 0, \end{cases}
\]
and define a modified functional \( \tilde{f} : W^{1,p}_0(\Omega) \to \mathbb{R} \) by putting

\[
\tilde{f}(u) = \int_\Omega \tilde{j}(x, u, \nabla u) \, dx - \frac{1}{p^*} \int_\Omega |u^+|^{p^*} \, dx - \int_\Omega \tilde{G}(x, u) \, dx,
\]
where \( \tilde{G}(x, s) = \int_0^s \tilde{g}(x, t) \, dt \). The Euler’s equation of \( \tilde{f} \), say \( \mathcal{C}_{\tilde{g}} \), is given by

\[
-\text{div} \left( \tilde{j}_\xi(x, u, \nabla u) \right) + \tilde{j}_s(x, u, \nabla u) = |u^+|^{p^*-2} u^+ + \tilde{g}(x, u) \text{ in } \Omega
\]
with \( u = 0 \) on \( \partial\Omega \).
Remark 3.1. Arguing as in [15, Lemma 1] one shows that if \( u \in W_0^{1,p}(\Omega) \) solves \( \tilde{g} \), then \( u \) solves \( g \). In particular, without loss of generality, from now on we will assume that
\[
\forall s \leq 0: \quad g(x, s) = 0, \quad j(x, s, \xi) = j(x, 0, \xi)
\]
for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^n \).

Remark 3.2. As pointed out in [6, 7], even if the functional \( f \) fails to be smooth it is possible to compute the directional derivatives along the bounded directions, i.e. for each \( u \in H_0^1(\Omega) \) one has
\[
f'(u)(v) = \int_{\Omega} j_{\xi}(x, u, \nabla u) \cdot \nabla v \, dx + \int_{\Omega} j_s(x, u, \nabla u) v \, dx - \int_{\Omega} (|u|^{p'-2}uv - g(x, u)v) \, dx
\]
for all \( v \in H_0^1 \cap L^\infty(\Omega) \).

Let us now prove that the concrete Palais–Smale sequences of \( f \) are bounded in \( W_0^{1,p}(\Omega) \). We will make a new choice of test function, which also removes some of the technicalities involved in [14].

Lemma 3.3. Let \( c \in \mathbb{R} \). Then each \((CPS)_c\)–sequence for \( f \) is bounded.

Proof. Let \( c \in \mathbb{R} \) and let \((u_h)\) be a \((CPS)_c\)–sequence for \( f \). In the notations of (13) one has \( \|u_h\|_{-1,p'} \to 0 \) as \( h \to +\infty \). It is easily verified that for each \( \alpha \in [p, p^*] \) there exists \( b_\alpha \in L^1(\Omega) \) with:
\[
g(x, s)s + |s|^{p'} \geq \alpha \left\{ G(x, s) + \frac{1}{p} |s|^{p'} \right\} - b_\alpha(x)
\]
a.e. in \( \Omega \) and for each \( s \in \mathbb{R} \). Let \( p < \alpha < p^* \) and \( M > 1 \) so that
\[
\alpha - \frac{M}{M-1} \gamma - \frac{M}{M-1} p > 0.
\]
Moreover, for \( k \geq 1 \) define a map \( \vartheta_k : \mathbb{R} \to \mathbb{R} \) by setting
\[
\vartheta_k(s) = \begin{cases} 
  s & \text{if } s \geq kM \\
  \frac{M}{M-1}s - \frac{M}{M-1}k & \text{if } k \leq s \leq kM \\
  0 & \text{if } -k \leq s < k \\
  \frac{M}{M-1}s + \frac{M}{M-1}k & \text{if } -kM \leq s < -k \\
  s & \text{if } s \leq -kM.
\end{cases}
\]
Since for each \( k \) we have \( f'(u_h)(\vartheta_k(u_h)) = o(1) \) as \( h \to +\infty \), there exists \( C_{k,M} > 0 \) such
that:
\[
\int_{\{u_h \geq kM\}} pj(x, u_h, \nabla u_h) \, dx + \frac{M}{M - 1} \int_{\{k \leq |u_h| \leq kM\}} pj(x, u_h, \nabla u_h) \, dx \\
+ \int_{\{u_h \geq kM\}} js(x, u_h, \nabla u_h) u_h \, dx + \frac{M}{M - 1} \int_{\{k \leq |u_h| \leq kM\}} js(x, u_h, \nabla u_h)(u_h \pm k) \, dx \\
= \int_{\{u_h \geq kM\}} g(x, u_h) u_h \, dx + \frac{M}{M - 1} \int_{\{k \leq |u_h| \leq kM\}} g(x, u_h)(u_h \pm k) \, dx \\
+ \int_{\{u_h \geq kM\}} |u_h|^p \, dx + \frac{M}{M - 1} \int_{\{k \leq |u_h| \leq kM\}} |u_h|^{p-2} u_h(u_h \pm k) \, dx + \langle w_h, \vartheta_k(u_h) \rangle \\
\geq \int_{\Omega} g(x, u_h) u_h \, dx - kM \int_{\{u_h \leq kM\}} |g(x, u_h)| \, dx + \frac{M}{M - 1} \int_{\{k \leq |u_h| \leq kM\}} g(x, u_h)(u_h \pm k) \, dx \\
- kM \int_{\{u_h \leq kM\}} |u_h|^{p-1} \, dx + \frac{M}{M - 1} \int_{\{k \leq |u_h| \leq kM\}} |u_h|^{p-2} u_h(u_h \pm k) \, dx + \langle w_h, \vartheta_k(u_h) \rangle \\
\geq \alpha \int_{\Omega} j(x, u_h, \nabla u_h) \, dx - \alpha f(u_h) - \int_{\Omega} b_\alpha(x) \, dx - C_{k,M} + \langle w_h, \vartheta_k(u_h) \rangle.
\]

On the other hand, by (5) and (4) one obtains
\[
\int_{\{u_h \geq \widetilde{k}\}} js(x, u_h, \nabla u_h) u_h \, dx \leq \gamma \int_{\{u_h \geq \widetilde{k}\}} j(x, u_h, \nabla u_h) \, dx, \tag{15}
\]
and
\[
- \widetilde{k} \int_{\{\widetilde{k} \leq u_h \leq \widetilde{k} M\}} js(x, u_h, \nabla u_h) \, dx \leq 0,
\]
\[
\widetilde{k} \int_{\{-\widetilde{k} M \leq u_h \leq -\widetilde{k}\}} js(x, u_h, \nabla u_h) \, dx \leq 0,
\]
for some \(\widetilde{k} \geq 1\) so that \(\widetilde{k} \geq \max\{R, R'\}\). Therefore, we find \(\widetilde{C}_{\tilde{k},M} > 0\) with
\[
\frac{\nu}{p} \left( \alpha - \frac{M}{M - 1} \gamma - \frac{M}{M - 1} p \right) \int_{\Omega} |\nabla u_h|^p \, dx \\
\leq \left( \alpha - \frac{M}{M - 1} \gamma - \frac{M}{M - 1} p \right) \int_{\Omega} j(x, u_h, \nabla u_h) \, dx \\
\leq \alpha f(u_h) + \int_{\Omega} b_\alpha(x) \, dx + \widetilde{C}_{\tilde{k},M} + \|w_h\|_{-1,p'} \|\vartheta_k(u_h)\|_{1,p}.
\]
Since \( f(u_h) \to c \) and \( w_h \to 0 \) in \( W^{-1,p'}(\Omega) \), the assertion follows.

**Remark 3.4.** It has to be pointed out that with the choice of test function \( \vartheta_k \) there is no need of using Lemma 3.3 in [14], which, though being interesting, involves lots of very technical computations.

**Lemma 3.5.** Let \( c \in \mathbb{R} \) and let \((u_h)\) be a \((CPS)_c\)-sequence for \( f \) such that \( u_h \to 0 \). Then for each \( \varepsilon > 0 \) and \( \rho > 0 \) we have
\[
\int_{\{\|u_h\| \leq \rho\}} j(x,u_h,\nabla u_h) \, dx \leq \varepsilon \int_{\{\|u_h\| > \rho\}} j(x,u_h,\nabla u_h) \, dx + o(1),
\]
for all \( h \in \mathbb{N} \).

**Proof.** It is a consequence of [14, Lemma 3.3] (See also [2]).

Let \( S \) denote the best Sobolev constant
\[
S = \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \|u\|_{p^*} = 1 \right\}.
\]

**Lemma 3.6.** Let \((u_h) \subset W_0^{1,p}(\Omega)\) be a concrete Palais–Smale sequence for \( f \) at level \( c \) with
\[
0 < c < \frac{1}{n} S^{n/p}.
\]
Assume that \( u_h \rightharpoonup u \). Then \( u \neq 0 \).

**Proof.** Assume by contradiction that \( u = 0 \). In particular, \( u \to 0 \) in \( L^s(\Omega) \) for each \( 1 \leq s < p^* \). Therefore, taking into account (6) and the \( p \)-homogeneity of \( j \) with respect to \( \xi \), from \( f'(u_h)(u_h) \to 0 \) we obtain
\[
\int_{\Omega} pj(x,u_h,\nabla u_h) \, dx + \int_{\Omega} js(x,u_h,\nabla u_h)u_h \, dx - \int_{\Omega} |u_h|^{p^*} \, dx = o(1),
\]
as \( h \to +\infty \). Let us now prove that for each \( \rho > 0 \)
\[
\lim_{h} \left| \int_{\{\|u_h\| \leq \rho\}} js(x,u_h,\nabla u_h)u_h \, dx \right| \leq \frac{C''}{\rho},
\]
for some \( C'' > 0 \). Indeed, since \( u_h \rightharpoonup 0 \), by Lemma 3.5 and (2), one has:
\[
\left| \int_{\{\|u_h\| \leq \rho\}} js(x,u_h,\nabla u_h)u_h \, dx \right|
\leq C \rho \int_{\{\|u_h\| \leq \rho\}} j(x,u_h,\nabla u_h) \, dx
\leq C \rho \varepsilon \int_{\{\|u_h\| \leq \rho\}} j(x,u_h,\nabla u_h) \, dx + o(1)
\leq C' \rho \varepsilon \int_{\Omega} |\nabla u_h|^p \, dx + o(1) \leq C'' \rho \varepsilon + o(1),
\]
for each \( \varrho > 0 \) and \( \varepsilon > 0 \) uniformly as \( h \to +\infty \). Then (17) follows by choosing \( \varepsilon = \frac{1}{\varrho^2} \). In particular, since condition (11) yields

\[
\lim_{\varrho \to +\infty} \int_{\{|u_h| > \varrho\}} j_s(x, u_h, \nabla u_h) u_h \, dx = 0,
\]

uniformly in \( h \in \mathbb{N} \), by combining (17) with (18), one gets

\[
\lim_h \int_{\Omega} j_s(x, u_h, \nabla u_h) u_h \, dx = 0.
\]

In a similar way, by (10), one shows that

\[
\int_{\Omega} j(x, u_h, \nabla u_h) \, dx = \frac{1}{p} \int_{\Omega} |\nabla u_h|^p \, dx + o(1)
\]

as \( h \to +\infty \). Therefore, by (16) one gets

\[
\|u_h\|_{1,p}^p - \|u_h\|_{p^*}^{p^*} = o(1),
\]

as \( h \to +\infty \). In particular, from the definition of \( S \), it holds

\[
\|u_h\|_{1,p}^p \left(1 - S^{-p/p}\|u_h\|_{1,p}^{p'-p}\right) \leq o(1),
\]

as \( h \to +\infty \). Since \( c > 0 \) it has to be

\[
\|u_h\|_{1,p}^p \geq S^{n/p} + o(1), \quad \|u_h\|_{p^*}^{p^*} \geq S^{n/p} + o(1),
\]

as \( h \to +\infty \). Hence, by (20) one deduces that

\[
f(u_h) = \frac{1}{n} \|u_h\|_{1,p}^p + \frac{1}{p^*}(\|u_h\|_{1,p}^p - \|u_h\|_{p^*}^{p^*}) + o(1) \geq \frac{1}{n} S^{n/p},
\]

contradicting the assumption. \[\square\]

4 Proof of Theorem 1.1

Let us consider the min–max class

\[
\Gamma = \{ \gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0, \; \gamma(1) = w \}
\]

with \( f(w) < 0 \) and

\[
\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f(\gamma(t)).
\]

Then, by the mountain pass theorem in its nonsmooth version (see [7]), one finds a Palais–Smale sequence for \( f \) at level \( \beta \). We have to prove that

\[
0 < \beta < \frac{1}{n} S^{n/p}.
\]
Consider the family of maps on \( \mathbb{R}^n \)

\[
T_{\delta,x_0}(x) = \frac{c_n \delta^{\frac{n-p}{p-1}}}{\left( \delta^{\frac{p}{p-1}} + |x - x_0|^{\frac{p}{p-1}} \right)^{\frac{n-p}{p}}} \tag{22}
\]

with \( \delta > 0 \) and \( x_0 \in \mathbb{R}^n \). \( T_{\delta,x_0} \) is a solution of \(-\Delta_p u = u^{p^*-1}\) on \( \mathbb{R}^n \). Taking a function \( \phi \in C_c^\infty(\Omega) \) with \( 0 \leq \phi \leq 1 \) and \( \phi = 1 \) in a neighbourhood of \( x_0 \) and setting \( \delta \phi = \phi T_{\delta,x_0} \), it results

\[
\|\delta \phi\|_{1,p}^p = S^{n/p} + o \left( \delta^{(n-p)/(p-1)} \right), \quad \|\delta \phi\|_{p^*}^p = S^{n/p} + o \left( \delta^{n/(p-1)} \right) \tag{23}
\]

as \( \delta \to 0 \), so that, as \( \delta \to 0 \),

\[
\frac{t_0^p}{p} \|\delta \phi\|_{1,p}^p - \frac{t_0^{p^*}}{p^*} \|\delta \phi\|_{p^*}^p \leq \frac{1}{n} S^{n/p} + o \left( \delta^{(n-p)/(p-1)} \right). \tag{24}
\]

To prove (21), it suffices to show that, for \( \delta > 0 \) small

\[
\sup_{t>0} f(t\delta \phi) < \frac{1}{n} S^{n/p}.
\]

Assume by contradiction that for each \( \delta > 0 \) there exists \( t_\delta > 0 \) with

\[
f(t_\delta \delta \phi) = \frac{t_\delta^p}{p} \|\delta \phi\|_{1,p}^p + t_\delta^p \int_\Omega \left\{ j(x, t_\delta \delta \phi, \nabla \delta \phi) - \frac{1}{p} |\nabla \delta \phi|^p \right\} dx \tag{25}
\]

\[
- \int_\Omega G(x, t_\delta \delta \phi) \, dx - \frac{t_\delta^{p^*}}{p^*} \|\delta \phi\|_{p^*}^p \geq \frac{1}{n} S^{n/p}
\]

In particular, the sequence \( (t_\delta) \) is bounded. Moreover, as proved in [3], by assumptions (8) if \( n < p^2 \) and (9) if \( n \geq p^2 \), there exists a function \( \tau : [0, 1] \to \mathbb{R} \) with \( \tau(\delta) \to +\infty \)

\[
\int_\Omega G(x, t_\delta \delta \phi) \, dx \geq \tau(\delta) \delta^{(n-p)/(p-1)}. \tag{26}
\]

as \( \delta \to 0 \). By (4) and (10) one also has

\[
\int_\Omega \left\{ j(x, t_\delta \delta \phi, \nabla \delta \phi) - \frac{1}{p} |\nabla \delta \phi|^p \right\} dx \leq 0 \tag{27}
\]

for each \( \delta > 0 \). By putting together (24), (25), (26), (27), one concludes

\[
f(t_\delta \delta \phi) \leq \frac{1}{n} S^{n/p} + (C - \tau(\delta)) \delta^{(n-p)/(p-1)}
\]

which contradicts (25) for \( \delta > 0 \) sufficiently small. \( \square \)
5 The compactness range

Let $\alpha \in [\gamma + p, p^*]$ be such that

$$g(x, s)s \geq \alpha G(x, s)$$

(28)

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$.

Assume now that (2), (3), (4), (5) with $R' = 0$, (6), (12) and (28) hold.

**Theorem 5.1.** The functional $f$ satisfies (CPS)$_c$ with

$$0 < c < \frac{p^* - \gamma - p}{p^* (\gamma + p)} (\nu S)^{\nu/p}$$

(29)

**Proof.** Let $(u_h)$ be a concrete Palais–Smale sequence for $f$ at level $c$. Since by Lemma 3.3 $(u_h)$ is bounded in $W^{1, p}_0(\Omega)$, up to a subsequence we have:

$$u_h \to u \text{ in } L^p(\Omega), \quad \nabla u_h \to \nabla u \text{ in } L^p(\Omega).$$

Moreover, as shown in [4], we also have:

$$\text{for a.e. } x \in \Omega : \quad \nabla u_h(x) \to \nabla u(x).$$

Arguing as in [14, Theorem 3.2] we get

$$\int_{\Omega} g(x, u) u \, dx + \|u\|_{p^*} = \int_{\Omega} j_{\xi}(x, u, \nabla u) \cdot \nabla u \, dx
\quad + \int_{\Omega} j_s(x, u, \nabla u) u \, dx.$$

This, following again [14, Theorem 3.2], yields the existence of $d \in \mathbb{R}$ with

$$\limsup_{h} \left\{ \int_{\Omega} j_{\xi}(x, u_h, \nabla u_h) \cdot \nabla u_h - \int_{\Omega} |u_h|^{p^*} \, dx \right\} \leq d
\quad \leq \int_{\Omega} j_{\xi}(x, u, \nabla u) \cdot \nabla u - \int_{\Omega} |u|^{p^*} \, dx.$$ (30)

Of course, we have:

$$\left\{ j_{\xi}(x, u_h, \nabla u_h) - j_{\xi}(x, u_h, \nabla (u_h - u)) \right\} \to j_{\xi}(x, u, \nabla u)$$

in $L^{p'}(\Omega)$. Let us note that it actually holds the strong limit

$$\left\{ j_{\xi}(x, u_h, \nabla u_h) - j_{\xi}(x, u_h, \nabla (u_h - u)) \right\} \to j_{\xi}(x, u, \nabla u)$$
in \(L^p(\Omega)\), since by (12) there exist \(\tau \in ]0, 1[\) and \(c > 0\) with:
\[
\begin{align*}
|j_\xi(x, u_h, \nabla u_h) - j_\xi(x, u_h, \nabla (u_h - u))| &
\leq |j_\xi(x, u_h, \nabla u_h + (\tau - 1)\nabla u)| |\nabla u| \\
&
\leq c|\nabla u_h|^{p-2}|\nabla u| + c|\nabla u|^{p-1}.
\end{align*}
\]
Therefore, it results
\[
\begin{align*}
\int_{\Omega} j_\xi(x, u_h, \nabla u_h) \cdot \nabla u_h \, dx &
= \int_{\Omega} j_\xi(x, u_h, \nabla (u_h - u)) \cdot \nabla u_h \, dx \\
+ \int_{\Omega} j_\xi(x, u, \nabla u) \cdot \nabla u_h \, dx + o(1) &
= \int_{\Omega} j_\xi(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) \, dx \\
+ \int_{\Omega} j_\xi(x, u, \nabla u) \cdot \nabla u \, dx + o(1),
\end{align*}
\]
as \(h \to +\infty\), namely
\[
\begin{align*}
\int_{\Omega} [j_\xi(x, u_h, \nabla u_h) \cdot \nabla u_h - j_\xi(x, u, \nabla u) \cdot \nabla u] \, dx \\
= \int_{\Omega} j_\xi(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) \, dx + o(1),
\end{align*}
\]
as \(h \to +\infty\). In a similar way, since there exists \(\tilde{c} > 0\) with
\[
\left| |u_h|^{p^*} - |u_h|^{p^*-p}|u_h - u|^p \right| \leq \tilde{c} \left[ |u_h|^{p^*-p}(|u_h|^{p-1} + |u|^{p-1}) \right] |u|,
\]
one obtains
\[
\left\{ |u_h|^{p^*} - |u_h|^{p^*-p}|u_h - u|^p \right\} \to |u|^{p^*} \quad \text{in} \quad L^1(\Omega).
\]
In particular, by combining (30), (31) and (32), it results:
\[
\limsup_{h} \int_{\Omega} \left[ j_\xi(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) \\
- |u_h|^{p^*-p}|u_h - u|^p \right] \, dx \leq 0.
\]
On the other hand, by H"older and Sobolev inequalities, we get:
\[
\begin{align*}
\int_{\Omega} \left[ j_\xi(x, u_h, \nabla (u_h - u)) \cdot \nabla (u_h - u) - |u_h|^{p^*-p}|u_h - u|^p \right] \, dx \\
\geq \nu \|\nabla (u_h - u)\|_p^p \frac{1}{S} \|u_h\|^{p^*-p}_p \|\nabla (u_h - u)\|_p^p \\
= \left\{ \nu - \frac{1}{S} \|u_h\|^{p^*-p}_p \right\} \|\nabla (u_h - u)\|_p^p,
\end{align*}
\]
which turns out to be coercive if:

$$\limsup_{h} \| u_h \|_{p^*} < (\nu S)^{n/p}. \quad (35)$$

Now, from $f(u_h) \to c$ we deduce

$$\begin{aligned}
(\gamma + p) \int_{\Omega} j(x, u_h, \nabla u_h) \, dx - \frac{\gamma + p}{p^*} \| u_h \|_{p^*}^p \\
= (\gamma + p) \int_{\Omega} G(x, u) \, dx + \gamma + p) c + o(1),
\end{aligned} \quad (36)$$

as $h \to +\infty$. By using (5), from $f'(u_h)(u_h) \to 0$ we obtain

$$\begin{aligned}
(\gamma + p) \int_{\Omega} j(x, u_h, \nabla u_h) \, dx - \| u_h \|_{p^*}^p \geq \int_{\Omega} g(x, u) \, dx + o(1),
\end{aligned} \quad (37)$$

as $h \to +\infty$. Therefore, by combining (36) with (37), one gets

$$\begin{aligned}
\frac{p^*- \gamma - p}{p^*} \| u_h \|_{p^*}^p \leq (\gamma + p) \int_{\Omega} G(x, u) \, dx \\
- \int_{\Omega} g(x, u) \, dx + (\gamma + p) c + o(1)
\end{aligned} \quad (38)$$

as $h \to +\infty$. Now, taking into account (28), we deduce that

$$\| u_h \|_{p^*}^p \leq \frac{p^*(\gamma + p)}{p^* - \gamma - p} c + o(1),$$

as $h \to +\infty$. In particular, condition (35) is fulfilled if

$$\frac{p^*(\gamma + p)}{p^* - \gamma - p} c < (\nu S)^{n/p}$$

which yields (29). By combining (33) and (34) we conclude the proof. \(\square\)

**Remark 5.2.** In the case $j_s = 0$ and $\nu = 1$, since $\gamma = 0$, (29) reduces to the well known range $0 < c < \frac{1}{n} S^{n/p}$.

**References**


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