Dynamics of Continuous, Discrete and Impulsive Systems
Series **A:** *Mathematical Analysis* **10 (2003)**, 193–208

ON A CLASS ON NONLINEAR ELLIPTIC BVP AT CRITICAL GROWTH

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Abstract. Existence and multiplicity of solutions for a general class of quasilinear elliptic problems at critical growth with perturbations of lower order are considered. Techniques of nonsmooth critical point theory are employed.

Keywords. nonlinear elliptic problems, critical exponent, Palais–Smale condition.
AMS subject classification: 35J20, 58E05, 74G35.

1 Introduction and main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $1 < p < n$. This paper is devoted to the existence and multiplicity of solutions in $W_0^{1,p}(\Omega)$ to the following class of nonlinear elliptic BVP, say $\mathscr{C}_g$, involving the critical Sobolev exponent $p^* = \frac{np}{n-p}$

$$\begin{cases} -\operatorname{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = |u|^{p^*-2}u + g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As a particular case, for $p < q < p^*$, we consider problems, say $\mathscr{C}_{\varepsilon,\lambda}$,

$$\begin{cases} -\operatorname{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = |u|^{p^*-2}u + \lambda|u|^{q-2}u + \varepsilon h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with $h \in L^{p'}(\Omega)$, $h \neq 0$, $\lambda > 0$ and $\varepsilon > 0$.

Motivations for investigating $\mathscr{C}_g$ and $\mathscr{C}_{\varepsilon,\lambda}$ come from various situations in geometry and physics which involve lack of compactness (see e.g. [5]). Since, as known, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ fails to be compact, one encounters serious difficulties in applying variational methods.

We refer the reader to [5] for the case $j = -\Delta$ and to [3, 9, 10] for the extension to degenerate operators ($j = -\Delta_p$, $p \neq 2$). For the existence of multiple solutions (two) for

[1] Partially supported by Ministero dell'Università e della Ricerca Scientifica e Tecnologica (40% – 1999) and by Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni.

problems $\mathscr{C}_{\varepsilon,\lambda}$, we refer the reader to [17] for $j = -\Delta$ and to [8] for $j = -\Delta_p$. In these cases the associated functional is smooth (C^1).

On the other hand, under natural assumptions on j, the functional associated to $\mathscr{C}_g$ and $\mathscr{C}_{\varepsilon,\lambda}$ is merely continuous (not even locally Lipschitz continuous) unless j does not depend on u or it is subjected to some restrictive growth conditions. Therefore, in general one also has lack of regularity and techniques of nonsmooth critical point theory have to be employed (see e.g. [6, 7] and references therein).

Quite recently, some existence results for problem

$$\begin{cases} -\operatorname{div}\,(j_\xi(x,u,\nabla u)) + j_s(x,u,\nabla u) = b(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

with b subcritical and superlinear have been considered in [1, 12] and e.g. in [6, 7, 14] via different points of view. It is therefore natural to wonder what happens when b reaches the critical growth and has some subcritical perturbation $g \neq 0$.

The first existence result in this framework was given in [2] for

$$j(x,s,\xi) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j, \quad b(x,u) = \lambda u + |u|^{2*-2}u,$$

provided that the a_{ij}s satisfy some suitable assumptions, including an asymptotic behaviour as s goes to $+\infty$ (cf. conditions (10) and (11)).

In view of this result, it is expected that under suitable assumptions on g and j problems $\mathscr{C}_g$ admit at least one nontrivial solution and problems $\mathscr{C}_{\varepsilon,\lambda}$ admit at least two nontrivial solutions for $\lambda > 0$ large and $\varepsilon > 0$ small (which depends on λ). The goal of this paper is precisely to prove these results thus extending the achievements of [3, 5, 8, 9, 10, 17] to this general and unified setting.

To carry on our analysis, we look for critical points (in a suitable sense) of the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by

$$f(u) = \int_\Omega j(x,u,\nabla u)\,dx - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,dx - \int_\Omega G(x,u)\,dx \tag{1}$$

where $G(x,s) = \int_0^s g(x,t)\,dt$ and $W_0^{1,p}(\Omega)$ is endowed with the standard norm $\|u\|_{1,p} = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$.

We assume that $j(x,s,\xi) : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is measurable in x for all $(s,\xi) \in \mathbb{R}\times\mathbb{R}^n$, of class C^1 in (s,ξ) and that $j(x,s,\cdot)$ is strictly convex and $p-$homogeneous with $j(x,s,0) = 0$. Moreover:

$(\mathscr{A}_1)$ there exist $\nu > 0$ and $c_1, c_2 > 0$ such that:

$$j(x,s,\xi) \geqslant \frac{\nu}{p}|\xi|^p, \quad |j_s(x,s,\xi)| \leqslant c_1|\xi|^p, \tag{2}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$|j_\xi(x,s,\xi)| \leqslant c_2 |\xi|^{p-1}, \tag{3}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

$(\mathscr{A}_2)$ there exist $R, R' > 0$ and $\gamma \in (0, p^* - p)$ such that:

$$|s| \geqslant R \implies j_s(x,s,\xi)s \geqslant 0, \tag{4}$$

$$|s| \geqslant R' \implies j_s(x,s,\xi)s \leqslant \gamma j(x,s,\xi), \tag{5}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

$(\mathscr{A}_3)$ Let λ_1 be the first eigenvalue of $-\Delta_p$ with homogeneous boundary conditions and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory map with $g(x,0) = 0$ and:

$$\forall \varepsilon > 0 \;\; \exists a_\varepsilon \in L^{\frac{np}{n(p-1)+p}}(\Omega) : \quad |g(x,s)| \leqslant a_\varepsilon(x) + \varepsilon |s|^{p^*-1}, \tag{6}$$

$$\limsup_{s \to 0} \frac{G(x,s)}{|s|^p} < \frac{\nu \lambda_1}{p}, \qquad G(x,s) \geqslant 0, \tag{7}$$

uniformly in Ω. Moreover, we assume that there exists a nonempty open set $\Omega_0 \subset \Omega$ such that:

- if $n < p^2$ (critical dimensions) :

$$\lim_{s \to +\infty} \frac{G(x,s)}{s^{p(np+p-2n)/(p-1)(n-p)}} = +\infty\,, \tag{8}$$

uniformly in Ω_0.

- if $n \geqslant p^2$: $\exists \mu > 0$, $\exists b > a$:

$$\forall s \in [a,b] : \; G(x,s) \geqslant \mu \tag{9}$$

for a.e. $x \in \Omega_0$ (μ sufficiently large in the case $n = p^2$).

Conditions (2), (3), (4) and (5) have already been considered e.g. in [1, 12, 14], while assumptions (6), (7), (8) and (9) can be found in [3]. Note that $g(x,u)$ is neither assumed to be positive nor homogeneous in u.

Assume now furthermore that (asymptotic behaviour):

$$\lim_{s \to +\infty} j(x,s,\xi) = \frac{1}{p} |\xi|^p, \tag{10}$$

$$\lim_{s \to +\infty} j_s(x,s,\xi)s = 0, \tag{11}$$

uniformly with respect to $x \in \Omega$ and to $\xi \in \mathbb{R}^n$ with $|\xi| \leqslant 1$. This means that there exist $\varepsilon_1 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon_2 : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$j(x,s,\xi) = \frac{1}{p}|\xi|^p + \varepsilon_1(x,s,\xi)|\xi|^p, \quad j_s(x,s,\xi)s = \varepsilon_2(x,s,\xi)|\xi|^p$$

where $\varepsilon_{1,2}(x,s,\xi) \to 0$ as $s \to +\infty$ uniformly for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$.

Under assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$, $(\mathscr{A}_3)$, (10) and (11), in Section 4 we will prove the following result.

Theorem 1.1. *$\mathscr{C}_g$ admits at least* <u>one nontrivial positive solution</u>.

This result extends the achievements of [2, 3] to a more general class of elliptic boundary value problems and gives a more complete picture of the results of [16].

Assume now that $\gamma \in (0, q-p)$ and $R' = 0$ in (5) and (12) holds. Moreover, let $j(x,s,\cdot)$ be of class C^2 and $b_1 > 0$ with

$$|j_{\xi\xi}(x,s,\xi)| \leqslant b_1|\xi|^{p-2} \tag{12}$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Under assumptions $(\mathscr{A}_1)$, $(\mathscr{A}_2)$ and (12), we have the following result.

Theorem 1.2. *$\mathscr{C}_{\varepsilon,\lambda}$ admits at least* <u>two nontrivial solutions</u> *provided that $\lambda > 0$ is sufficiently large and $\varepsilon > 0$ is sufficiently small (depending on λ).*

For the full proof of this result, we refer the reader to [16]. In this paper, we prefer to prove in Section 5 a general version of the compactness theorem.
This result extends the achievements of [8] to a more general class of elliptic boundary value problems. We stress that we proved our result without any use of concentration–compactness techniques [11]. From this point of view, our approach seems to be simpler and more direct.
Assume finally that Ω is star–shaped, $h = 0$, $\lambda \leqslant 0$ and

$$p^* j_x(x,s,\xi) \cdot x - n j_s(x,s,\xi) s \geqslant 0$$

a.e. in Ω and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Then from the general variational identity of Pucci–Serrin [13], we derive the following result.

Theorem 1.3. *$\mathscr{C}_{\varepsilon,\lambda}$ admits* <u>no nontrivial solution</u> *$u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.*

For the proof, we refer the reader to [16, Corollary 6.2].

2 Recalls of nonsmooth analysis

Let (X,d) be a metric space and let $f : X \to \mathbb{R}$ be a continuous function.

Definition 2.1. *(see [7]). For every $u \in X$ we denote by $|df|(u)$ the supremum of σs in $[0,+\infty[$ such that there exist $\delta > 0$ and a continuous map $\mathscr{H} : B_\delta(u) \times [0,\delta] \to X$ such that*

$$\forall v \in B_\delta(u),\ \forall t \in [0,\delta] : \qquad d(\mathscr{H}(v,t),v) \leqslant t\,,$$
$$\forall v \in B_\delta(u),\ \forall t \in [0,\delta] : \qquad f(\mathscr{H}(v,t)) \leqslant f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called the weak slope *of f at u.*

If f is of class C^1 and X is normed, $|df|(u) = \|df(u)\|_X$ for each $u \in X$.

Definition 2.2. *We say that $u \in X$ is a* critical point *for f if $|df|(u) = 0$. Let $c \in \mathbb{R}$. We say that (u_h) is a* Palais–Smale sequence *for f at level c ($(PS)_c-$sequence) if $f(u_h) \to c$ and $|df|(u_h) \to 0$. We say that f satisfies the* Palais–Smale condition *at level c if every $(PS)_c-$sequence for f admits a convergent subsequence.*

Definition 2.3. *A sequence $(u_h) \subset W_0^{1,p}(\Omega)$ is said to be a* concrete Palais Smale sequence *at level $c \in \mathbb{R}$ ($(CPS)_c-$sequence) for f, if $f(u_h) \to c$,*

$$-\mathrm{div}\ (j_\xi(x, u_h, \nabla u_h)) + j_s(x, u_h, \nabla u_h) \in W^{-1,p'}(\Omega),$$

eventually as $h \to +\infty$ and

$$-\mathrm{div}\ (j_\xi(x, u_h, \nabla u_h)) + j_s(x, u_h, \nabla u_h) - |u_h|^{p^*-2}u_h - g(x, u_h) \to 0$$

strongly in $W^{-1,p'}(\Omega)$. We say that f satisfies the concrete Palais–Smale condition *at level c ($(CPS)_c$ in short), if every $(CPS)_c-$sequence for f admits a strongly convergent subsequence.*

Definition 2.4. *We say that u is a* weak solution *to $\mathscr{C}_g$ if $u \in W_0^{1,p}(\Omega)$ and*

$$-\mathrm{div}\ (j_\xi(x, u, \nabla u)) + j_s(x, u, \nabla u) = |u|^{p^*-2}u + g(x, u)$$

in distributional sense.

Proposition 2.5. *Let $u \in W_0^{1,p}(\Omega)$ be such that $|df|(u) < +\infty$. Then*

$$w_u := -\mathrm{div}\ (j_\xi(x, u, \nabla u)) + j_s(x, u, \nabla u) - |u|^{p^*-2}u - g(x, u) \tag{13}$$

belongs to $W^{-1,p'}(\Omega)$ and $\|w_u\|_{-1,p'} \leqslant |df|(u)$.

In particular, if u is a critical point of f then u is a weak solution to $\mathscr{C}_g$. Finally, it is readily seen by the above Proposition that if f satisfies $(CPS)_c$, then is satisfies (PS).

3 Existence of one nontrivial solution

Let us set for a.e. $x \in \Omega$ and for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$

$$\widetilde{j}(x, s, \xi) = \begin{cases} j(x, s, \xi) & \text{if } s \geqslant 0 \\ j(x, 0, \xi) & \text{if } s < 0, \end{cases} \qquad \widetilde{g}(x, s) = \begin{cases} g(x, s) & \text{if } s \geqslant 0 \\ 0 & \text{if } s < 0, \end{cases}$$

and define a modified functional $\widetilde{f} : W_0^{1,p}(\Omega) \to \mathbb{R}$ by putting

$$\widetilde{f}(u) = \int_\Omega \widetilde{j}(x, u, \nabla u)\, dx - \frac{1}{p^*} \int_\Omega |u^+|^{p^*}\, dx - \int_\Omega \widetilde{G}(x, u)\, dx, \tag{14}$$

where $\widetilde{G}(x, s) = \int_0^s \widetilde{g}(x, t)\, dt$. The Euler's equation of $\widetilde{f}$, say $\widetilde{\mathscr{C}_g}$, is given by

$$-\mathrm{div}\,(\widetilde{j}_\xi(x, u, \nabla u)) + \widetilde{j}_s(x, u, \nabla u) = |u^+|^{p^*-2}u^+ + \widetilde{g}(x, u) \text{ in } \Omega$$

with $u = 0$ on $\partial\Omega$.

Remark 3.1. *Arguing as in [15, Lemma 1] one shows that if $u \in W_0^{1,p}(\Omega)$ solves of $\widetilde{\mathscr{C}}_g$, then u solves $\mathscr{C}_g$. In particular, without loss of generality, from now on we will assume that*

$$\forall s \leqslant 0: \quad g(x,s)=0, \quad j(x,s,\xi)=j(x,0,\xi)$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

Remark 3.2. *As pointed out in [6, 7], even if the functional f fails to be smooth it is possible to compute the directional derivatives along the bounded directions, i.e. for each $u \in H_0^1(\Omega)$ one has*

$$f'(u)(v) = \int_\Omega j_\xi(x,u,\nabla u)\cdot \nabla v\,dx + \int_\Omega j_s(x,u,\nabla u)v\,dx - \int_\Omega \left(|u|^{p^*-2}uv - g(x,u)v\right)\,dx.$$

for all $v \in H_0^1 \cap L^\infty(\Omega)$.

Let us now prove that the concrete Palais–Smale sequences of f are bounded in $W_0^{1,p}(\Omega)$. We will make a new choice of test function, which also removes some of the technicalities involved in [14].

Lemma 3.3. *Let $c \in \mathbb{R}$. Then each $(CPS)_c$−sequence for f is bounded.*

Proof. Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$−sequence for f. In the notations of (13) one has $\|w_h\|_{-1,p'} \to 0$ as $h \to +\infty$. It is easily verified that for each $\alpha \in [p,p^*[$ there exists $b_\alpha \in L^1(\Omega)$ with:

$$g(x,s)s + |s|^{p^*} \geqslant \alpha\left\{G(x,s) + \frac{1}{p^*}|s|^{p^*}\right\} - b_\alpha(x)$$

a.e. in Ω and for each $s \in \mathbb{R}$. Let $p < \alpha < p^*$ and $M > 1$ so that

$$\alpha - \frac{M}{M-1}\gamma - \frac{M}{M-1}p > 0.$$

Moreover, for $k \geqslant 1$ define a map $\vartheta_k : \mathbb{R} \to \mathbb{R}$ by setting

$$\vartheta_k(s) = \begin{cases} s & \text{if } s \geqslant kM \\ \frac{M}{M-1}s - \frac{M}{M-1}k & \text{if } k \leqslant s \leqslant kM \\ 0 & \text{if } -k \leqslant s \leqslant k \\ \frac{M}{M-1}s + \frac{M}{M-1}k & \text{if } -kM \leqslant s \leqslant -k \\ s & \text{if } s \leqslant -kM. \end{cases}$$

Since for each k we have $f'(u_h)(\vartheta_k(u_h)) = o(1)$ as $h \to +\infty$, there exists $C_{k,M} > 0$ such that:

$$
\begin{aligned}
&\int_{\{|u_h|\geqslant kM\}} pj(x,u_h,\nabla u_h)\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}} pj(x,u_h,\nabla u_h)\,dx\\
&+\int_{\{|u_h|\geqslant kM\}} j_s(x,u_h,\nabla u_h)u_h\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}} j_s(x,u_h,\nabla u_h)(u_h\pm k)\,dx\\
&=\int_{\{|u_h|\geqslant kM\}} g(x,u_h)u_h\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}} g(x,u_h)(u_h\pm k)\,dx\\
&+\int_{\{|u_h|\geqslant kM\}} |u_h|^{p^*}\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}} |u_h|^{p^*-2}u_h(u_h\pm k)\,dx+\langle w_h,\vartheta_k(u_h)\rangle\\
&\geqslant\int_\Omega g(x,u_h)u_h\,dx-kM\int_{\{|u_h|\leqslant kM\}}|g(x,u_h)|\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}} g(x,u_h)(u_h\pm k)\,dx\\
&+\int_\Omega |u_h|^{p^*}\,dx-kM\int_{\{|u_h|\leqslant kM\}}|u_h|^{p^*-1}\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}}|u_h|^{p^*-2}u_h(u_h\pm k)\,dx\\
&+\langle w_h,\vartheta_k(u_h)\rangle\geqslant\alpha\left[\int_\Omega G(x,u_h)\,dx+\frac{1}{p^*}\int_\Omega|u_h|^{p^*}\,dx\right]-\int_\Omega b_\alpha(x)\,dx\\
&-kM\int_{\{|u_h|\leqslant kM\}}|g(x,u_h)|\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}} g(x,u_h)(u_h\pm k)\,dx\\
&-kM\int_{\{|u_h|\leqslant kM\}}|u_h|^{p^*-1}\,dx+\frac{M}{M-1}\int_{\{k\leqslant|u_h|\leqslant kM\}}|u_h|^{p^*-2}u_h(u_h\pm k)\,dx+\langle w_h,\vartheta_k(u_h)\rangle\\
&\geqslant\alpha\int_\Omega j(x,u_h,\nabla u_h)\,dx-\alpha f(u_h)-\int_\Omega b_\alpha(x)\,dx-C_{k,M}+\langle w_h,\vartheta_k(u_h)\rangle.
\end{aligned}
$$

On the other hand, by (5) and (4) one obtains

$$
\int_{\{|u_h|\geqslant\overline{k}\}} j_s(x,u_h,\nabla u_h)u_h\,dx\leqslant\gamma\int_{\{|u_h|\geqslant\overline{k}\}} j(x,u_h,\nabla u_h)\,dx, \tag{15}
$$

and

$$
-\overline{k}\int_{\{\overline{k}\leqslant u_h\leqslant\overline{k}M\}} j_s(x,u_h,\nabla u_h)\,dx\leqslant 0,
$$

$$
\overline{k}\int_{\{-\overline{k}M\leqslant u_h\leqslant-\overline{k}\}} j_s(x,u_h,\nabla u_h)\,dx\leqslant 0,
$$

for some $\overline{k}\geqslant 1$ so that $\overline{k}\geqslant\max\{R,R'\}$. Therefore, we find $\widetilde{C}_{\overline{k},M}>0$ with

$$
\begin{aligned}
&\frac{\nu}{p}\left(\alpha-\frac{M}{M-1}\gamma-\frac{M}{M-1}p\right)\int_\Omega|\nabla u_h|^p\,dx\\
&\leqslant\left(\alpha-\frac{M}{M-1}\gamma-\frac{M}{M-1}p\right)\int_\Omega j(x,u_h,\nabla u_h)\,dx\\
&\leqslant\alpha f(u_h)+\int_\Omega b_\alpha(x)\,dx+\widetilde{C}_{\overline{k},M}+\|w_h\|_{-1,p'}\|\vartheta_{\overline{k}}(u_h)\|_{1,p}.
\end{aligned}
$$

Since $f(u_h) \to c$ and $w_h \to 0$ in $W^{-1,p'}(\Omega)$, the assertion follows. $\square$

Remark 3.4. *It has to be pointed out that with the choice of test function ϑ_k there is no need of using Lemma 3.3 in [14], which, though being interesting, involves lots of very technical computations.*

Lemma 3.5. *Let $c \in \mathbb{R}$ and let (u_h) be a $(CPS)_c$−sequence for f such that $u_h \rightharpoonup 0$. Then for each $\varepsilon > 0$ and $\varrho > 0$ we have*

$$\int_{\{|u_h| \leqslant \varrho\}} j(x, u_h, \nabla u_h)\, dx \leqslant \varepsilon \int_{\{|u_h| > \varrho\}} j(x, u_h, \nabla u_h)\, dx + o(1),$$

for all $h \in \mathbb{N}$.

Proof. It is a consequence of [14, Lemma 3.3] (See also [2]). $\square$

Let S denote the best Sobolev constant

$$S = \inf \left\{ \|\nabla u\|_p^p : \; u \in W_0^{1,p}(\Omega), \; \|u\|_{p^*} = 1 \right\}.$$

Lemma 3.6. *Let $(u_h) \subset W_0^{1,p}(\Omega)$ be a concrete Palais–Smale sequence for f at level c with*

$$0 < c < \frac{1}{n} S^{n/p}.$$

Assume that $u_h \rightharpoonup u$. Then $u \neq 0$.

Proof. Assume by contradiction that $u = 0$. In particular, $u \to 0$ in $L^s(\Omega)$ for each $1 \leqslant s < p^*$. Therefore, taking into account (6) and the p−homogeneity of j with respect to ξ, from $f'(u_h)(u_h) \to 0$ we obtain

$$\int_\Omega pj(x, u_h, \nabla u_h)\, dx + \int_\Omega j_s(x, u_h, \nabla u_h) u_h\, dx - \int_\Omega |u_h|^{p^*}\, dx = o(1)\,, \tag{16}$$

as $h \to +\infty$. Let us now prove that for each $\varrho > 0$

$$\lim_h \left| \int_{\{|u_h| \leqslant \varrho\}} j_s(x, u_h, \nabla u_h) u_h\, dx \right| \leqslant \frac{C''}{\varrho}, \tag{17}$$

for some $C'' > 0$. Indeed, since $u_h \rightharpoonup 0$, by Lemma 3.5 and (2), one has:

$$\begin{aligned}
&\left| \int_{\{|u_h| \leqslant \varrho\}} j_s(x, u_h, \nabla u_h) u_h\, dx \right| \\
&\leqslant C\varrho \int_{\{|u_h| \leqslant \varrho\}} j(x, u_h, \nabla u_h)\, dx \\
&\leqslant C\varrho\varepsilon \int_{\{|u_h| > \varrho\}} j(x, u_h, \nabla u_h)\, dx + o(1) \\
&\leqslant C'\varrho\varepsilon \int_\Omega |\nabla u_h|^p\, dx + o(1) \leqslant C''\varrho\varepsilon + o(1),
\end{aligned}$$

for each $\varrho > 0$ and $\varepsilon > 0$ uniformly as $h \to +\infty$. Then (17) follows by choosing $\varepsilon = \frac{1}{\varrho^2}$. In particular, since condition (11) yields

$$\lim_{\varrho \to +\infty} \int_{\{|u_h| > \varrho\}} j_s(x, u_h, \nabla u_h) u_h \, dx = 0, \tag{18}$$

uniformly in $h \in \mathbb{N}$, by combining (17) with (18), one gets

$$\lim_h \int_\Omega j_s(x, u_h, \nabla u_h) u_h \, dx = 0. \tag{19}$$

In a similar way, by (10), one shows that

$$\int_\Omega j(x, u_h, \nabla u_h) \, dx = \frac{1}{p} \int_\Omega |\nabla u_h|^p \, dx + o(1) \tag{20}$$

as $h \to +\infty$. Therefore, by (16) one gets

$$\|u_h\|_{1,p}^p - \|u_h\|_{p^*}^{p^*} = o(1),$$

as $h \to +\infty$. In particular, from the definition of S, it holds

$$\|u_h\|_{1,p}^p \left(1 - S^{-p^*/p} \|u_h\|_{1,p}^{p^*-p}\right) \leqslant o(1)\,,$$

as $h \to +\infty$. Since $c > 0$ it has to be

$$\|u_h\|_{1,p}^p \geqslant S^{n/p} + o(1), \qquad \|u_h\|_{p^*}^{p^*} \geqslant S^{n/p} + o(1),$$

as $h \to +\infty$. Hence, by (20) one deduces that

$$f(u_h) = \frac{1}{n} \|u_h\|_{1,p}^p + \frac{1}{p^*} (\|u_h\|_{1,p}^p - \|u_h\|_{p^*}^{p^*}) + o(1) \geqslant \frac{1}{n} S^{n/p}\,,$$

contradicting the assumption. □

4 Proof of Theorem 1.1

Let us consider the min–max class

$$\varGamma = \left\{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \ \gamma(0) = 0, \ \gamma(1) = w\right\}$$

with $f(w) < 0$ and

$$\beta = \inf_{\gamma \in \varGamma} \max_{t \in [0,1]} f(\gamma(t)).$$

Then, by the mountain pass theorem in its nonsmooth version (see [7]), one finds a Palais–Smale sequence for f at level β. We have to prove that

$$0 < \beta < \frac{1}{n} S^{n/p}. \tag{21}$$

Consider the family of maps on $\mathbb{R}^n$

$$T_{\delta,x_0}(x) = \frac{c_n \delta^{\frac{n-p}{p(p-1)}}}{\left(\delta^{\frac{p}{p-1}} + |x-x_0|^{\frac{p}{p-1}}\right)^{\frac{n-p}{p}}} \tag{22}$$

with $\delta > 0$ and $x_0 \in \mathbb{R}^n$. T_{δ,x_0} is a solution of $-\Delta_p u = u^{p^*-1}$ on $\mathbb{R}^n$. Taking a function $\phi \in C_c^\infty(\Omega)$ with $0 \leqslant \phi \leqslant 1$ and $\phi = 1$ in a neighbourhood of x_0 and setting $v_\delta = \phi T_{\delta,x_0}$, it results

$$\|v_\delta\|_{1,p}^p = S^{n/p} + o\left(\delta^{(n-p)/(p-1)}\right), \quad \|v_\delta\|_{p^*}^{p^*} = S^{n/p} + o\left(\delta^{n/(p-1)}\right) \tag{23}$$

as $\delta \to 0$, so that, as $\delta \to 0$,

$$\frac{t_\delta^p}{p}\|v_\delta\|_{1,p}^p - \frac{t_\delta^{p^*}}{p^*}\|v_\delta\|_{p^*}^{p^*} \leqslant \frac{1}{n}S^{n/p} + o\left(\delta^{(n-p)/(p-1)}\right). \tag{24}$$

To prove (21), it suffices to show that, for $\delta > 0$ small

$$\sup_{t \geqslant 0} f(tv_\delta) < \frac{1}{n}S^{n/p}.$$

Assume by contradiction that for each $\delta > 0$ there exists $t_\delta > 0$ with

$$\begin{aligned} f(t_\delta v_\delta) = \frac{t_\delta^p}{p}\|v_\delta\|_{1,p}^p + t_\delta^p \int_\Omega \left\{ j(x, t_\delta v_\delta, \nabla v_\delta) - \frac{1}{p}|\nabla v_\delta|^p \right\} dx \\ - \int_\Omega G(x, t_\delta v_\delta)\, dx - \frac{t_\delta^{p^*}}{p^*}\|v_\delta\|_{p^*}^{p^*} \geqslant \frac{1}{n}S^{n/p} \end{aligned} \tag{25}$$

In particular, the sequence (t_δ) is bounded. Moreover, as proved in [3], by assumptions (8) if $n < p^2$ and (9) if $n \geqslant p^2$, there exists a function $\tau : [0,1] \to \mathbb{R}$ with $\tau(\delta) \to +\infty$ and

$$\int_\Omega G(x, t_\delta v_\delta)\, dx \geqslant \tau(\delta)\delta^{(n-p)/(p-1)}. \tag{26}$$

as $\delta \to 0$. By (4) and (10) one also has

$$\int_\Omega \left\{ j(x, t_\delta v_\delta, \nabla v_\delta) - \frac{1}{p}|\nabla v_\delta|^p \right\} dx \leqslant 0 \tag{27}$$

for each $\delta > 0$. By putting together (24), (25), (26), (27), one concludes

$$f(t_\delta v_\delta) \leqslant \frac{1}{n}S^{n/p} + (C - \tau(\delta))\delta^{(n-p)/(p-1)}$$

which contradicts (25) for $\delta > 0$ sufficiently small. □

5 The compactness range

Let $\alpha \in [\gamma + p, p^*[$ be such that

$$g(x,s)s \geqslant \alpha G(x,s) \tag{28}$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$.

Assume now that (2), (3), (4), (5) with $R' = 0$, (6), (12) and (28) hold.

Theorem 5.1. *The functional f satisfies $(CPS)_c$ with*

$$0 < c < \frac{p^* - \gamma - p}{p^*(\gamma + p)} (\nu S)^{n/p} \tag{29}$$

Proof. Let (u_h) be a concrete Palais–Smale sequence for f at level c. Since by Lemma 3.3 (u_h) is bounded in $W_0^{1,p}(\Omega)$, up to a subsequence we have:

$$u_h \to u \quad \text{in } L^p(\Omega)\,, \qquad \nabla u_h \rightharpoonup \nabla u \quad \text{in } L^p(\Omega).$$

Moreover, as shown in [4], we also have:

$$\text{for a.e. } x \in \Omega: \quad \nabla u_h(x) \to \nabla u(x).$$

Arguing as in [14, Theorem 3.2] we get

$$\begin{aligned}\int_\Omega g(x,u)u\,dx + \|u\|_{p^*}^{p^*} = &\int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u\,dx \\ &+ \int_\Omega j_s(x,u,\nabla u)u\,dx.\end{aligned}$$

This, following again [14, Theorem 3.2], yields the existence of $d \in \mathbb{R}$ with

$$\begin{aligned}&\limsup_h \left\{\int_\Omega j_\xi(x,u_h,\nabla u_h)\cdot\nabla u_h - \int_\Omega |u_h|^{p^*}\,dx\right\} \leqslant d \\ &\leqslant \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u - \int_\Omega |u|^{p^*}\,dx.\end{aligned} \tag{30}$$

Of course, we have:

$$\left\{j_\xi(x,u_h,\nabla u_h) - j_\xi(x,u_h,\nabla(u_h-u))\right\} \rightharpoonup j_\xi(x,u,\nabla u)$$

in $L^{p'}(\Omega)$. Let us note that it actually holds the strong limit

$$\left\{j_\xi(x,u_h,\nabla u_h) - j_\xi(x,u_h,\nabla(u_h-u))\right\} \to j_\xi(x,u,\nabla u)$$

in $L^{p'}(\Omega)$, since by (12) there exist $\tau \in]0,1[$ and $c > 0$ with:

$$\begin{aligned}
&|j_\xi(x,u_h,\nabla u_h) - j_\xi(x,u_h,\nabla(u_h-u))| \\
&\leqslant |j_{\xi\xi}(x,u_h,\nabla u_h + (\tau-1)\nabla u)|\,|\nabla u| \\
&\leqslant c|\nabla u_h|^{p-2}|\nabla u| + c|\nabla u|^{p-1}.
\end{aligned}$$

Therefore, it results

$$\begin{aligned}
&\int_\Omega j_\xi(x,u_h,\nabla u_h)\cdot\nabla u_h\,dx = \int_\Omega j_\xi(x,u_h,\nabla(u_h-u))\cdot\nabla u_h\,dx \\
&+ \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u_h\,dx + o(1) = \int_\Omega j_\xi(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u)\,dx \\
&+ \int_\Omega j_\xi(x,u,\nabla u)\cdot\nabla u\,dx + o(1)\,,
\end{aligned}$$

as $h \to +\infty$, namely

$$\begin{aligned}
&\int_\Omega \left[j_\xi(x,u_h,\nabla u_h)\cdot\nabla u_h - j_\xi(x,u,\nabla u)\cdot\nabla u\right]\,dx \\
&= \int_\Omega j_\xi(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u)\,dx + o(1)\,,
\end{aligned} \tag{31}$$

as $h \to +\infty$. In a similar way, since there exists $\widetilde{c} > 0$ with

$$\left||u_h|^{p^*} - |u_h|^{p^*-p}|u_h-u|^p\right| \leqslant \widetilde{c}\left[|u_h|^{p^*-p}(|u_h|^{p-1}+|u|^{p-1})\right]|u|,$$

one obtains

$$\left\{|u_h|^{p^*} - |u_h|^{p^*-p}|u_h-u|^p\right\} \to |u|^{p^*} \quad\text{in}\quad L^1(\Omega). \tag{32}$$

In particular, by combining (30), (31) and (32), it results:

$$\begin{aligned}
&\limsup_h \int_\Omega \Big[j_\xi(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) \\
&- |u_h|^{p^*-p}|u_h-u|^p\Big]\,dx \leqslant 0.
\end{aligned} \tag{33}$$

On the other hand, by Hölder and Sobolev inequalities, we get:

$$\begin{aligned}
&\int_\Omega \left[j_\xi(x,u_h,\nabla(u_h-u))\cdot\nabla(u_h-u) - |u_h|^{p^*-p}|u_h-u|^p\right]\,dx \\
&\geqslant \nu\|\nabla(u_h-u)\|_p^p - \frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\|\nabla(u_h-u)\|_p^p \\
&= \left\{\nu - \frac{1}{S}\|u_h\|_{p^*}^{p^*-p}\right\}\|\nabla(u_h-u)\|_p^p\,,
\end{aligned} \tag{34}$$

which turns out to be coercive if:

$$\limsup_h \|u_h\|_{p^*}^{p^*} < (\nu S)^{n/p}. \tag{35}$$

Now, from $f(u_h) \to c$ we deduce

$$\begin{aligned}(\gamma+p)\int_\Omega j(x,u_h,\nabla u_h)\,dx - \frac{\gamma+p}{p^*}\|u_h\|_{p^*}^{p^*} \\ = (\gamma+p)\int_\Omega G(x,u)\,dx + (\gamma+p)c + o(1)\,,\end{aligned} \tag{36}$$

as $h \to +\infty$. By using (5), from $f'(u_h)(u_h) \to 0$ we obtain

$$(\gamma+p)\int_\Omega j(x,u_h,\nabla u_h)\,dx - \|u_h\|_{p^*}^{p^*} \geqslant \int_\Omega g(x,u)u\,dx + o(1)\,, \tag{37}$$

as $h \to +\infty$. Therefore, by combining (36) with (37), one gets

$$\begin{aligned}\frac{p^*-\gamma-p}{p^*}\|u_h\|_{p^*}^{p^*} \leqslant (\gamma+p)\int_\Omega G(x,u)\,dx \\ - \int_\Omega g(x,u)u\,dx + (\gamma+p)c + o(1)\end{aligned} \tag{38}$$

as $h \to +\infty$. Now, taking into account (28), we deduce that

$$\|u_h\|_{p^*}^{p^*} \leqslant \frac{p^*(\gamma+p)}{p^*-\gamma-p}c + o(1),$$

as $h \to +\infty$. In particular, condition (35) is fulfilled if

$$\frac{p^*(\gamma+p)}{p^*-\gamma-p}c < (\nu S)^{n/p}$$

which yields (29). By combining (33) and (34) we conclude the proof. □

Remark 5.2. *In the case* $j_s = 0$ *and* $\nu = 1$, *since* $\gamma = 0$, (29) *reduces to the well known range* $0 < c < \frac{1}{n}S^{n/p}$.

References

[1] D. Arcoya, L. Boccardo, Critical points for multiple integrals of the calculus of variations, *Arch. Ration. Mech. Anal.* **134** (1996), 249–274.

[2] G. Arioli, F. Gazzola, Quasilinear elliptic equations at critical growth, *NoDEA, Nonlinear Differential Equations Appl.* **5** (1998), 83–97.

[3] G. Arioli, F. Gazzola, Some results on $p-$Laplace equations with a critical growth term, *Differential Integral Equations* **11** (1998), 311–326.

[4] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, *Nonlinear Anal.* **19** (1992), 581–597.

[5] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponent, *Comm. Pure Appl. Math.* **36** (1983), 437–477.

[6] A. Canino, Multiplicity of solutions for quasilinear elliptic equations, *Topol. Methods Nonlinear Anal.* **6** (1995), 357–370.

[7] A. Canino, M. Degiovanni, Nonsmooth critical point theory and quasilinear elliptic equations, *Topological Methods in Differential Equations and Inclusions*, 1–50 – A. Granas, M. Frigon, G. Sabidussi Eds. – Montreal (1994), NATO ASI Series – Kluwer A.P. (1995).

[8] J. Chabrowski, On multiple solutions for the non–homogeneous $p-$Laplacian with a critical Sobolev exponent, *Differential Integral Equations* (4) **8** (1995), 705–716.

[9] J. Garcia Azorero, I. Peral Alonso, Existence and nonuniqueness for the $p-$Laplacian, *Comm. Partial Differential Equations* **12** (1987), 1389–1430.

[10] J. Garcia Azorero, I. Peral Alonso, Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term, *Trans. Amer. Math. Soc.* **323** (1991), 887–895.

[11] P.L. Lions, The concentration-compactness principle in the calculus of variations. The limit case. II, *Rev. Mat. Iberoamericana* **1** (1985), 45–121.

[12] B. Pellacci, Critical points for non-differentiable functionals, *Boll. Un. Mat. Ital. B* **11** (1997), 733–749.

[13] P. Pucci, J. Serrin, A general variational identity, *Indiana Univ. Math. J.* **35** (1986), 681–703.

[14] M. Squassina, Existence of weak solutions to general Euler's equations via nonsmooth critical point theory, *Ann. Fac. Sci. Toulouse Math.* (1) **9** (2000), 113–131.

[15] M. Squassina, On the existence of positive entire solutions of nonlinear elliptic problems, *Topol. Methods Nonlinear Anal.* **17** (2001), 23–39.

[16] M. Squassina, Two solutions for inhomogeneous fully nonlinear elliptic equations at critical growth, *NoDEA, Nonlinear Differential Equations Appl.* in press (2002).

[17] G. Tarantello, On nonhomogeneous elliptic equations involving critical Sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire* (3) **9** (1992), 281–304.

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