ON THE EXISTENCE OF SOLUTIONS TO A FOURTH-ORDER QUASILINEAR RESONANT PROBLEM

SHIBO LIU AND MARCO SQUASSINA

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By means of Morse theory we prove the existence of a nontrivial solution to a superlinear p-harmonic elliptic problem with Navier boundary conditions having a linking structure around the origin. Moreover, in case of both resonance near zero and nonresonance at $+\infty$ the existence of two nontrivial solutions is shown.

1. Introduction and main results

Let $p>1$ and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain with $n \geqslant 2p+1$. We are concerned with the existence of nontrivial solutions to the p-harmonic equation

$$\Delta\big(|\Delta u|^{p-2}\Delta u\big) = g(x,u) \quad \text{in } \Omega \tag{1.1}$$

with Navier boundary conditions

$$u = \Delta u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for some $C>0$,

$$\big|g(x,s)\big| \leqslant C\big(1+|s|^{q-1}\big) \tag{1.3}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, being $1 \leqslant q < p_*$ and $p_* = np/(n-2p)$.

It is well known that the functional $\Phi : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to \mathbb{R}$

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p\,dx - \int_\Omega G(x,u)\,dx, \tag{1.4}$$

with $G(x,s) = \int_0^s g(x,t)\,dt$, is of class C^1 and

$$\big\langle \Phi'(u),\varphi\big\rangle = \int_\Omega |\Delta u|^{p-2}\Delta u\,\Delta\varphi\,dx - \int_\Omega g(x,u)\varphi\,dx \tag{1.5}$$

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for each $\varphi \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Moreover, the critical points of Φ are weak solutions for (1.1). Notice that for the eigenvalue problem

$$\Delta\left(|\Delta u|^{p-2}\Delta u\right) = \lambda |u|^{p-2}u \quad \text{in } \Omega \tag{1.6}$$

with boundary data (1.2), as for the p-Laplacian eigenvalue problem with Dirichlet boundary data,

$$\lambda_n = \inf_{A\in\Gamma_n}\sup_{u\in A}\int_\Omega |\Delta u|^p\,dx, \qquad n = 1,2,\dots \tag{1.7}$$

is the sequence of eigenvalues, where

$$\Gamma_n = \left\{A \subseteq W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\setminus\{0\} : A = -A,\ \gamma(A) \geqslant n\right\}, \tag{1.8}$$

being $\gamma(A)$ the Krasnoselski's genus of the set A. This follows by the Ljusternik-Schnirelman theory for C^1-manifolds proved in [13] applied to the functional

$$\begin{gathered} J|_{\mathcal{M}}(u) = \int_\Omega |\Delta u|^p\,dx, \\ \mathcal{M} = \left\{u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) : \int_\Omega |u|^p\,dx = 1\right\}, \end{gathered} \tag{1.9}$$

since $\mathcal{M}$ is a C^1-manifold with tangent space

$$T_u\mathcal{M} = \left\{w \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) : \int_\Omega |u|^{p-2}uw\,dx = 0\right\}. \tag{1.10}$$

The next remark is the starting point of our paper.

Remark 1.1. It has been recently proved by Drábek and Ôtani [4] that (1.6) with boundary data (1.2) has the least eigenvalue

$$\lambda_1(p) = \inf\left\{\int_\Omega |\Delta u|^p\,dx : u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega),\ \|u\|_p^p = 1\right\} \tag{1.11}$$

which is simple, positive, and isolated in the sense that the solutions of (1.6) with $\lambda = \lambda_1(p)$ form a one-dimensional linear space spanned by a positive eigenfunction $\phi_1(p)$ associated with $\lambda_1(p)$ and there exists $\delta > 0$ so that $(\lambda_1(p), \lambda_1(p)+\delta)$ does not contain other eigenvalues. The situation is actually more involved with Dirichlet boundary conditions

$$u = \nabla u = 0 \quad \text{on } \partial\Omega \tag{1.12}$$

and, to our knowledge, it is not clear whether the first eigenspace has the previous good properties; the fact is that while Navier boundary conditions allow to reduce the fourth-order problem into a system of two second-order problems, Dirichlet boundary conditions do not. Some pathologies are indeed known, for instance, the first eigenfunction of $\Delta^2 u = \lambda u$ with boundary data (1.12) may change sign [12].

Remark 1.2. Let $V = \operatorname{span}\{\phi_1\}$ be the eigenspace associated with λ_1, where $\|\phi_1\|_{2,p} = 1$. Taking a subspace $W \subset W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ complementing V, that is, $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) = V \oplus W$, there exists $\hat{\lambda} > \lambda_1$ with

$$\int_\Omega |\Delta u|^p \, dx \geqslant \hat{\lambda} \int_\Omega |u|^p \, dx \tag{1.13}$$

for each $u \in W$ (in case $p = 2$, one may take $\hat{\lambda} = \lambda_2$).

We may now assume the following conditions:

($\mathcal{H}_1$) there exist $R > 0$ and $\bar{\lambda} \in]\lambda_1, \hat{\lambda}[$ such that

$$|s| \leqslant R \Longrightarrow \lambda_1 |s|^p \leqslant pG(x,s) \leqslant \bar{\lambda}|s|^p, \tag{1.14}$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$;

($\mathcal{H}_2$) there exist $\vartheta > p$ and $M > 0$ such that

$$|s| \geqslant M \Longrightarrow 0 < \vartheta G(x,s) \leqslant sg(x,s), \tag{1.15}$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$.

Assumption ($\mathcal{H}_1$) corresponds to a resonance condition around the origin while ($\mathcal{H}_2$) is the standard condition of Ambrosetti-Rabinowitz type.

Theorem 1.3. *Assume that conditions ($\mathcal{H}_1$) and ($\mathcal{H}_2$) hold. Then problem (1.1) with boundary conditions (1.2) admits a nontrivial solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.*

Now replace ($\mathcal{H}_2$) with a nonresonance condition at $+\infty$.

Theorem 1.4. *Assume that condition ($\mathcal{H}_1$) holds and that for a.e. $x \in \Omega$*

$$\lim_{|s| \to +\infty} \frac{pG(x,s)}{|s|^p} < \lambda_1. \tag{1.16}$$

Then problem (1.1) with boundary conditions (1.2) admits two nontrivial solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.

We use variational methods to prove Theorems 1.3 and 1.4. Usually, one uses a minimax type argument of mountain pass type to prove the existence of solutions of equations with a variational structure. However, it seems difficult to use minimax theorems in our situation. Thus we will adopt an approach based on Morse theory. Notice that there were a few works using Morse theory to treat p-Laplacian problems with Dirichlet boundary conditions (see [9] and the references therein). Moreover, to the authors' knowledge, (1.1) has a very poor literature; the only papers in which a p-harmonic equation is mentioned are [1, Section 8] and [4].

The existence of multiple solutions depends mainly on the behaviour of $G(x,s)$ near 0 and at $+\infty$. Without the above resonant or nonresonant conditions to obtain multiple solutions seems hard even in the semilinear case $p=2$.

Remark 1.5. Arguing as in [9], it is possible to prove Theorem 1.4 by replacing assumption (1.16) with the following conditions:

$$\lim_{|s|\to+\infty}\frac{pG(x,s)}{|s|^p}=\lambda_1,\qquad \lim_{|s|\to+\infty}\{g(x,s)s-pG(x,s)\}=+\infty \tag{1.17}$$

for a.e. $x\in\Omega$ (resonance condition at $+\infty$).

Remark 1.6. The existence of solutions $u\in W_0^{2,p}(\Omega)$ of the quasilinear problem

$$\begin{aligned} \Delta\left(|\Delta u|^{p-2}\Delta u\right)&=g(x,u) &&\text{in } \Omega,\\ u=\nabla u&=0 &&\text{on } \partial\Omega \end{aligned} \tag{1.18}$$

under the previous assumptions $(\mathscr{H}_j)$ is, to our knowledge, an open problem.

2. Proofs of Theorems 1.3 and 1.4

In this section, we give the proof of our main results. It is readily seen that

$$\|u\|_{2,p}=\left(\int_\Omega|\Delta u|^p\,dx\right)^{1/p} \tag{2.1}$$

is an equivalent norm of the standard space norm of $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. For Φ a continuously Fréchet differentiable map, let Φ' denote its Fréchet derivative.

Lemma 2.1. *The functional Φ satisfies the Palais-Smale condition.*

Proof. Let $(u_h)\subset W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ be such that $|\Phi(u_h)|\leqslant B$, for some $B>0$ and $\Phi'(u_h)\to 0$ as $h\to+\infty$. Let $d=\sup_{h\geqslant 0}\Phi(u_h)$. Then we have

$$\begin{aligned} \vartheta d+\left\|u_h\right\|_{2,p} &\geqslant \vartheta\Phi(u_h)-\langle\Phi'(u_h),u_h\rangle\\ &=\left(\frac{\vartheta}{p}-1\right)\|u_h\|_{2,p}^p-\int_{\{|u_h|\geqslant M\}}\left[\vartheta G(x,u_h)-g(x,u_h)u_h\right]dx\\ &\quad-\int_{\{|u_h|\leqslant M\}}\left[\vartheta G(x,u_h)-g(x,u_h)u_h\right]dx\\ &\geqslant\left(\frac{\vartheta}{p}-1\right)\|u_h\|_{2,p}^p-\int_{\{|u_h|\leqslant M\}}\left[\vartheta G(x,u_h)-g(x,u_h)u_h\right]dx\\ &\geqslant\left(\frac{\vartheta}{p}-1\right)\|u_h\|_{2,p}^p-D, \end{aligned} \tag{2.2}$$

for some $D\in\mathbb{R}$. Thus (u_h) is bounded and, up to a subsequence, we may assume that $u_h\rightharpoonup u$ is, for some u, in $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Since the embedding

$W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\hookrightarrow L^q(\Omega)$ is compact, then a standard argument shows that $u_h\to u$ strongly and the proof is complete. □

Now recall the notion of "Local Linking," which was initially introduced by Liu and Li [8] and has been used in a vast amount of literature (cf. [2, 5, 6, 11]).

Definition 2.2. Let E be a real Banach space such that $E=V\oplus W$, where V and W are closed subspaces of E. Let $\Phi: E\to\mathbb{R}$ be a C^1-functional. We say that Φ has a local linking near the origin 0 (with respect to the decomposition $E=V\oplus W$), if there exists $\varrho>0$ such that

$$\begin{aligned} &u\in V:\|u\|\leqslant\varrho\Longrightarrow\Phi(u)\leqslant 0,\\ &u\in W: 0<\|u\|\leqslant\varrho\Longrightarrow\Phi(u)>0. \end{aligned}\tag{2.3}$$

We now show that our functional Φ has a local linking near the origin with respect to the space decomposition $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)=V\oplus W$, according to Remark 1.2.

Lemma 2.3. *There exists $\varrho>0$ such that conditions (2.3) hold with respect to the decomposition $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)=V\oplus W$.*

Proof. For $u\in V$, the condition $\|u\|_{2,p}\leqslant\varrho$ implies $u(x)\leqslant R$ for a.e. $x\in\Omega$ if $\varrho>0$ is small enough, being $R>0$ as in assumption $(\mathcal{H}_1)$. Thus for $u\in V$,

$$\begin{aligned}\Phi(u)&=\frac{1}{p}\int_\Omega|\Delta u|^p\,dx-\int_\Omega G(x,u)\,dx\\ &=\frac{\lambda_1}{p}\int_\Omega|u|^p\,dx-\int_\Omega G(x,u)\,dx=\int_{\{|u|\leqslant R\}}\left[\frac{\lambda_1}{p}|u|^p-G(x,u)\right]dx\leqslant 0\end{aligned}\tag{2.4}$$

provided that $\|u\|_{2,p}\leqslant\varrho$ and ϱ is small.

To prove the second assertion, take $u\in W$. In view of (1.3) and (1.13) we have

$$\begin{aligned}\Phi(u)&=\frac{1}{p}\int_\Omega|\Delta u|^p\,dx-\int_\Omega G(x,u)\,dx\\ &=\frac{1}{p}\int_\Omega\left(|\Delta u|^p-\bar\lambda|u|^p\right)dx\\ &\quad-\left(\int_{\{|u|\leqslant R\}}+\int_{\{|u|\geqslant R\}}\right)\left(G(x,u)-\frac{\bar\lambda}{p}|u|^p\right)dx\\ &\geqslant\frac{1}{p}\left(1-\frac{\bar\lambda}{\hat\lambda}\right)\|u\|_{2,p}^p-c\int_\Omega|u|^s\,dx\geqslant\frac{1}{p}\left(1-\frac{\bar\lambda}{\hat\lambda}\right)\|u\|_{2,p}^p-C\|u\|_{2,p}^s,\end{aligned}\tag{2.5}$$

where $p<s\leqslant p_*$ and c, C are positive constants. Since $s>p$, it follows that $\Phi(u)>0$ for $\varrho>0$ sufficiently small. □

Assume that u is an isolated critical point of Φ such that $\Phi(u) = c$. We define the *critical group* of Φ at u by setting for each $q \in \mathbb{Z}$

$$C_q(\Phi, u) = H_q\left(\Phi_c, \Phi_c \setminus \{u\}\right), \tag{2.6}$$

being $H_q(X, Y)$ the qth homology group of the topological pair (X, Y) over the ring $\mathbb{Z}$ and Φ_c the c-sublevel of Φ. For the detail of Morse theory and critical groups, we refer the reader to [3].

Since $\dim V = 1 < +\infty$, by combining Lemma 2.3 and [7, Theorem 2.1], we obtain the following result.

Lemma 2.4. *The point* 0 *is a critical point of* Φ *and* $C_1(\Phi, 0) \neq \{0\}$.

We now investigate the behavior of Φ near infinity.

Lemma 2.5. *There exists a constant* $A > 0$ *such that*

$$a < -A \Longrightarrow \Phi_a \simeq S^\infty, \tag{2.7}$$

where $S^\infty = \{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \|u\|_{2,p} = 1\}$.

Proof. By integrating inequality (1.15), we obtain a constant $C_1 > 0$ with

$$|s| \geqslant M \Longrightarrow G(x, s) \geqslant C_1 |s|^\vartheta \tag{2.8}$$

a.e. in Ω and for each $s \in \mathbb{R}$. Thus, for $u \in S^\infty$, we have $\Phi(tu) \to -\infty$, as t goes to $+\infty$. Set

$$A = \left(1 + \frac{1}{p}\right) M \mathscr{L}^n(\Omega) \max_{\bar{\Omega} \times [-M, M]} |g(x, s)| + 1, \tag{2.9}$$

being $\mathscr{L}^n$ the Lebesgue measure. As in the proof of [10, Lemma 2.4] we obtain

$$\begin{aligned} \int_\Omega G(x, u)\, dx - \frac{1}{p} \int_\Omega g(x, u) u\, dx \\ \leqslant \left(\frac{1}{\vartheta} - \frac{1}{p}\right) \int_{\{|u| \geqslant M\}} g(x, u) u\, dx + A - 1. \end{aligned} \tag{2.10}$$

For $a < -A$ and

$$\Phi(tu) = \frac{|t|^p}{p} - \int_\Omega G(x, tu)\, dx \leqslant a \quad (u \in S^\infty), \tag{2.11}$$

in view of (2.8) and (2.10), arguing as in the proof of [10, Lemma 2.4],

$$\frac{d}{dt} \Phi(tu) < 0. \tag{2.12}$$

By the implicit function theorem, there is a unique $T \in C(S^\infty, \mathbb{R})$ such that

$$\forall u \in S^\infty, \quad \Phi\big(T(u)u\big) = a. \tag{2.13}$$

For $u \neq 0$, set $\tilde{T}(u) = (1/\|u\|_{2,p})T(u/\|u\|_{2,p})$. Then $\tilde{T} \in C(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\}, \mathbb{R})$ and

$$\forall u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\}, \quad \Phi\big(\tilde{T}(u)u\big) = a. \tag{2.14}$$

We define now a functional $\hat{T} : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\} \to \mathbb{R}$ by setting

$$\hat{T}(u) = \begin{cases} \tilde{T}(u) & \text{if } \Phi(u) \geqslant a, \\ 1 & \text{if } \Phi(u) \leqslant a. \end{cases} \tag{2.15}$$

Since $\Phi(u) = a$ implies $\tilde{T}(u) = 1$, we conclude that

$$\hat{T} \in C\big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\}, \mathbb{R}\big). \tag{2.16}$$

Finally, let $\eta : [0,1] \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\} \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\}$,

$$\eta(s,u) = (1-s)u + s\hat{T}(u)u. \tag{2.17}$$

It results that η is a strong deformation retract from $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\}$ to Φ_a. Thus $\Phi_a \simeq W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\} \simeq S^\infty$. □

Remark 2.6. A result similar to Lemma 2.5 has been proved for the Laplacian $-\Delta$ in [3, 14], under the additional conditions

$$g \in C^1(\Omega \times \mathbb{R}, \mathbb{R}), \quad g_t(x,0) = \left.\frac{\partial g(x,t)}{\partial t}\right|_{t=0} = 0. \tag{2.18}$$

We recall the following topological result due to Perera [11].

Lemma 2.7. *Let $Y \subset B \subset A \subset X$ be topological spaces and $q \in \mathbb{Z}$. If*

$$H_q(A,B) \neq \{0\}, \qquad H_q(X,Y) = \{0\}, \tag{2.19}$$

then it results that

$$H_{q+1}(X,A) \neq \{0\} \qquad \text{or} \qquad H_{q-1}(B,Y) \neq \{0\}. \tag{2.20}$$

Proof of Theorem 1.3. By Lemma 2.1, Φ satisfies the Palais-Smale condition. Note that $\Phi(0) = 0$, by [3, Chapter I, Theorem 4.2], there exists $\varepsilon > 0$ with

$$H_1\big(\Phi_\varepsilon, \Phi_{-\varepsilon}\big) = C_1(\Phi, 0) \neq \{0\}. \tag{2.21}$$

If A is as in Lemma 2.5, for $a < -A$ we have $\Phi_a \simeq S^\infty$, which yields

$$H_1\big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \Phi_a\big) = H_1\big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), S^\infty\big) = \{0\}. \tag{2.22}$$

Therefore, being $\Phi_a \subset \Phi_{-\varepsilon} \subset \Phi_\varepsilon$, Lemma 2.7 yields

$$H_2\big(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \Phi_\varepsilon\big) \neq \{0\} \qquad \text{or} \qquad H_0\big(\Phi_{-\varepsilon}, \Phi_a\big) \neq \{0\}. \tag{2.23}$$

It follows that Φ has a critical point u for which

$$\Phi(u) > \varepsilon \qquad \text{or} \qquad -\varepsilon > \Phi(u) > a. \tag{2.24}$$

Therefore, $u \neq 0$ and (1.1), (1.2) possess a nontrivial solution. □

Recall from [9] the following three-critical point theorem.

Lemma 2.8. *Let X be a real Banach space and let $\Phi \in C^1(X, \mathbb{R})$ be bounded from below and satisfying the Palais-Smale condition. Assume that Φ has a critical point u which is homologically nontrivial, that is, $C_j(\Phi, u) \neq \{0\}$ for some j, and it is not a minimizer for Φ. Then Φ admits at least three critical points.*

Proof of Theorem 1.4. By Lemma 2.8, taking into account Lemma 2.4, it suffices to show that Φ is bounded from below. Indeed, by (1.16) there exist $\varepsilon > 0$ small and $C > 0$ such that

$$G(x,s) \leqslant \frac{\lambda_1 - \varepsilon}{p}|s|^p + C \tag{2.25}$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$. This, by (1.11), immediately yields

$$\begin{aligned} \Phi(u) &\geqslant \frac{1}{p}\|u\|_{2,p}^p - \frac{1}{p}(\lambda_1 - \varepsilon)\|u\|_p^p - C\mathcal{L}^n(\Omega) \\ &\geqslant \frac{1}{p}\left(1 - \frac{\lambda_1 - \varepsilon}{\lambda_1}\right)\|u\|_{2,p}^p - C\mathcal{L}^n(\Omega) \longrightarrow +\infty \end{aligned} \tag{2.26}$$

as $\|u\|_{2,p} \to +\infty$. Then Φ is coercive and satisfies the Palais-Smale condition. In particular Lemma 2.8 provides the existence of at least two nontrivial critical points of Φ. □

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Shibo Liu: Institute of Mathematics, Academy of Mathematics and Systems Sciences, Academia Sinica, Beijing 100080, China
E-mail address: liusb@amss.ac.cn

Marco Squassina: Dipartimento di Matematica, Università Cattolica S.C., Via Musei 41, 25121 Brescia, Italy
E-mail address: squassin@dmf.unicatt.it