Multiple critical points for perturbed symmetric functionals associated with quasilinear elliptic problems

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Abstract

By means of nonsmooth critical point theory, we prove existence of infinitely many weak solutions for a class of perturbed symmetric quasilinear elliptic equations.

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1 Introduction

The main goal of this paper is to extend to the quasilinear case the existence results known since 1980 for the semilinear scalar problem

\[
\begin{cases}
- \sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x,u) + \varphi & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with \(g\) is superlinear and odd in \(u\), \(\varphi \in L^2(\Omega)\) and \(\Omega\) is a bounded domain in \(\mathbb{R}^n\).

This problem has been deeply studied in [2], [10], [15], [17] and [19] by means of the variational techniques developed by Bahri, Berestycki, Rabinowitz and Struwe in the early eighties. We remark that, around 1990, Bahri and P. L. Lions improved in [3] and [4] the previous results via a technique based on

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Morse theory. See also [20] for further improvements by means of a completely different method devised by P. Bolle.

On the other hand, since 1994, several efforts have been devoted to study existence for quasilinear problems of the type

\[
\begin{cases}
- \sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2} \sum_{i,j=1}^{n} D_ja_{ij}(x,u)D_iuD_ju = g(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(1)

We refer the reader to [5], [6], [7] and [18] for the study of multiplicity of solutions of problem (1) and furthermore to [1], [14] and [16] for an even more general framework.

The functional \( f_0 : H_0^1(\Omega) \to \mathbb{R} \) associated with (1) is given by

\[
f_0(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju \, dx - \int_{\Omega} G(x,u) \, dx.
\]

We stress that \( f_0 \) is not even locally Lipschitz unless the \( a_{ij} \)'s do not depend on \( u \).

Consequently, techniques of nonsmooth critical point theory have to be applied. We refer the reader to [7], [8], [9], [11] and [12] for the abstract theory that we shall need in the following.

It seems now natural to ask whether also in a quasilinear setting the multiplicity of solutions is stable under large \( L^2 \)-perturbations.

In this paper we want to investigate the effects of destroying the symmetry of (1) and show that for each \( \varphi \in L^2(\Omega) \) the perturbed equation

\[- \sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2} \sum_{i,j=1}^{n} D_ja_{ij}(x,u)D_iuD_ju = g(x,u) + \varphi \text{ in } \Omega \]  

(2)

still has infinitely many weak solutions.

Therefore, we shall work on the functional

\[
f_\varphi(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju \, dx - \int_{\Omega} G(x,u) \, dx - \int_{\Omega} \varphi u \, dx.
\]

In the next, \( \Omega \) will denote an open and bounded subset of \( \mathbb{R}^n \) with \( n \geq 3 \). Moreover, we shall consider the following assumptions:
(\(H_1\)) each \(a_{ij}(x, s)\) is measurable in \(x\) for each \(s \in \mathbb{R}\) and of class \(C^1\) in \(s\) for a.e. \(x \in \Omega\) with \(a_{ij}(x, s) = a_{ji}(x, s), a_{ij}(\cdot, \cdot) \in L^\infty(\Omega \times \mathbb{R})\) and \(D_s a_{ij}(\cdot, \cdot) \in L^\infty(\Omega \times \mathbb{R})\). Moreover, there exist \(\nu > 0\) and \(R > 0\) such that

\[
\sum_{i,j=1}^n a_{ij}(x, s)\xi_i \xi_j \geq \nu |\xi|^2
\]

\[|s| \geq R \implies \sum_{i,j=1}^n s D_s a_{ij}(x, s)\xi_i \xi_j \geq 0,
\]

for a.e. \(x \in \Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\);

(\(H_2\)) \(G(x, s)\) is measurable in \(x\) for all \(s \in \mathbb{R}\), of class \(C^1\) in \(s\) a.e. in \(\Omega\) with \(G(x, 0) = 0\) and \(g(x, s) = D_s G(x, s)\). Moreover, there exist \(q > 2\) and \(R' > 0\) with

\[|s| \geq R' \implies 0 < qG(x, s) \leq g(x, s)s,\]

for a.e. \(x \in \Omega\) and all \(s \in \mathbb{R}\);

(\(H_3\)) there exists \(\gamma \in ]0, q - 2[\) such that :

\[|s| \geq R' \implies \sum_{i,j=1}^n s D_s a_{ij}(x, s)\xi_i \xi_j \leq \gamma \sum_{i,j=1}^n a_{ij}(x, s)\xi_i \xi_j,
\]

for a.e. \(x \in \Omega\) and for all \((s, \xi) \in \mathbb{R} \times \mathbb{R}^n\).

Under the previous assumptions, the following is our main result.

**Theorem 1.1** Assume that there exists \(\sigma \in ]1, \frac{qn+2(q-1)}{qn-2(q-1)}[\) such that

\[|g(x, s)| \leq a + b|s|^{\sigma}
\]

with \(a, b \in \mathbb{R}\) and that for a.e. \(x \in \Omega\) and for each \(s \in \mathbb{R}\)

\[a_{ij}(x, -s) = a_{ij}(x, s), \quad g(x, -s) = -g(x, s).
\]

Then there exists a sequence \((u_h)\) of weak solutions to (2) with \(f_\phi(u_h) \to +\infty\).
Since for each $q > 2$ and $n \geq 3$ we have
\[
\frac{qn + (q - 1)(n + 2)}{qn + (q - 1)(n - 2)} < \frac{qn + 2(q - 1)}{qn - 2(q - 1)},
\]
this result relaxes Theorem 1.1 of [13] which was proven for
\[
\sigma \in \left[1, \frac{qn + (q - 1)(n + 2)}{qn + (q - 1)(n - 2)} \right].
\]
This because we used in Lemma 4.2 the sharp estimate from below on the growth of the critical values shown by K. Tanaka in 1989 via Morse theory [19] which improves the direct estimate of P. H. Rabinowitz of 1981 based on the Gagliardo–Niremberg inequality.

We point out that we assumed (3) and (5) only for large values of $|s|$, while in [13], dealing with systems, we requested conditions (3) and (5) to hold for each $s \in \mathbb{R}$.

In the next result we allow a more general class of perturbations.

**Theorem 1.2** Assume that there exists $\sigma \in [1, 2]$ and $\sigma' \in [0, q - 1]$ with
\[
|g(x, s)| \leq a + b|s|^\sigma, \quad |\varphi(x, s)| \leq c + d|s|^\sigma',
\]
where $\varphi \in C(\Omega \times \mathbb{R})$ and $\frac{2(\sigma + 1)}{n(\sigma - 1)} > \frac{q}{q - \sigma' - 1}$. Assume further that:
\[
a_{ij}(x, -s) = a_{ij}(x, s), \quad g(x, -s) = -g(x, s).
\]
Then there exists a sequence $(u_n)$ of weak solutions of the problem
\[
- \sum_{i,j=1}^{n} D_j(a_{ij}(x, u)D_i u) + \frac{1}{2} \sum_{i,j=1}^{n} D_ja_{ij}(x, u)D_i u D_j u = g(x, u) + \varphi(x, u) \quad \text{in } \Omega
\]
with $u = 0$ on $\partial \Omega$, such that $f_\Phi(u_n) \to +\infty$ where:
\[
f_\Phi(u) = \frac{1}{2} \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x, u)D_i u D_j u \, dx - \int_{\Omega} G(x, u) \, dx - \int_{\Omega} \Phi(x, u) \, dx,
\]
and $D_s \Phi(x, s) = \varphi(x, s)$ for each $x \in \Omega$ and all $s \in \mathbb{R}$.

These theorems extend the results of [2], [10], [15] and [17] to the quasilinear case.
2 Perturbation of even functionals

If \( \varphi \neq 0 \), clearly \( f_{\varphi} \) is not even. Note that by (4) we find \( c_1, c_2, c_3 > 0 \) such that:

\[
\frac{1}{q} (g(x, s)s + c_1) \geq G(x, s) + c_2 \geq c_3 |s|^q ,
\]

for a.e. \( x \in \Omega \) and each \( s \in \mathbb{R} \).

**Lemma 2.1** Assume that \( u \) is a weak solution to (2). Then it results

\[
\int_\Omega (G(x, u) + c_2) \, dx \leq \sigma \left( f_{\varphi}^2(u) + 1 \right)^{\frac{1}{2}},
\]

for some \( \sigma > 0 \) depending on \( \|\varphi\|_2 \).

**Proof.** Let us set \( C = \|D_s a_{ij}\|_{L^\infty(\Omega \times \mathbb{R})} \). If we choose \( \gamma' \in ]\gamma, q - 2[ \) and \( \varepsilon > 0 \) in such a way that

\[
\frac{nC R^2 \varepsilon}{\nu} \leq \gamma' - \gamma,
\]

using [7, Theorem 2.2.9] and working as in the proof of [7, Lemma 2.3.2], we get

\[
\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x, u) D_i u D_j u \, dx \leq
\]

\[
\leq \gamma' \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u) D_i u D_j u \, dx + M_{R', \varepsilon} .
\]

(7)

Therefore, we deduce that:

\[
f_{\varphi}(u) = f_{\varphi}(u) - \frac{1}{2} f'_{\varphi}(u)(u) =
\]

\[
= \int_\Omega \left[ \frac{1}{2} g(x, u) u - G(x, u) - \frac{1}{2} \varphi u \right] \, dx - \frac{1}{4} \int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x, u) D_i u D_j u \, dx \geq
\]

\[
\geq \left( \frac{1}{2} - \frac{1}{q} \right) \int_\Omega (g(x, u) u + c_1) \, dx - \frac{\|\varphi\|_2 \|u\|_2}{2} +
\]

\[
- \frac{\gamma'}{4} \int_\Omega \sum_{i,j=1}^n a_{ij}(x, u) D_i u D_j u \, dx - c_4 \geq
\]

\[
\geq \left( \frac{q}{2} - 1 - \frac{\gamma'}{2} \right) \int_\Omega (G(x, u) + c_2) \, dx - \frac{\gamma'}{2} f_{\varphi}(u) - \delta \|u\|_q^q - \beta(\delta) \|\varphi\|_2^{q'} - c_5
\]

with \( \delta \to 0 \) and \( \beta(\delta) \to +\infty \). Choosing \( \delta > 0 \) small enough, by (6) we have:
\[ \sigma_\delta f_\varphi(u) \geq \int_\Omega (G(x,u) + c_2) \, dx - c_6, \]

where \( \sigma_\delta = \frac{2 + \gamma}{q - 2 - \gamma - 2\delta} \). The assertion follows as in [15, Lemma 1.8].

Let us now define \( \chi \in C^\infty(\mathbb{R}) \) by setting \( \chi = 1 \) for \( s \leq 1 \), \( \chi = 0 \) for \( s \geq 2 \) and \( -2 < \chi' < 0 \) when \( 1 < s < 2 \), and let us set:

\[ \phi(u) = 2\sigma \left(f_\varphi^2(u) + 1\right)^{\frac{1}{2}}, \quad \psi(u) = \chi \left(\phi(u)^{-1} \int_\Omega (G(x,u) + c_2) \, dx\right) \]

for each \( u \in H^1_0(\Omega) \). Finally, we define the modified functional by setting:

\[ \tilde{f}_\varphi(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx - \int_\Omega G(x,u) \, dx - \psi(u) \int_\Omega \varphi \, u \, dx. \tag{8} \]

The Euler's equation associated with (8) is given by:

\[ \begin{cases}
- \sum_{i,j=1}^n D_j(a_{ij}(x,u)D_i u) + \frac{1}{2} \sum_{i,j=1}^n D_s a_{ij}(x,u)D_i u D_j u = \bar{g}(x,u) & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial \Omega,
\end{cases} \tag{9} \]

where we have set:

\[ \bar{g}(x,u) = g(x,u) + \psi(u)\varphi + \psi'(u) \int_\Omega \varphi \, u \, dx. \]

We remark that by Lemma 2.1, if \( u \) solves (2), then \( \psi(u) = 1 \) and \( \tilde{f}_\varphi(u) = f_\varphi(u) \). In the next result, we measure the defect of symmetry of \( \tilde{f}_\varphi \).

**Lemma 2.2** There exists \( \beta > 0 \) depending on \( \|\varphi\|_2 \) such that

\[ |\tilde{f}_\varphi(u) - \tilde{f}_\varphi(-u)| \leq \beta \left\{ |\tilde{f}_\varphi(u)|^{\frac{1}{2}} + 1 \right\} \]

for each \( u \in H^1_0(\Omega) \).

**Proof.** See [13, Lemma 2.2].

**Theorem 2.3** There exists \( \tilde{M} > 0 \) such that if \( u \) is a weak solution of (9) such that \( \tilde{f}_\varphi(u) \geq \tilde{M} \) then \( u \) is a weak solution to (2) and \( \tilde{f}_\varphi(u) = f_\varphi(u) \).
Proof. For the complete proof, see [13, Theorem 2.3]. Let us give a brief sketch. Standard computations yield:

\[ \psi'(u)(u) = \chi'(\vartheta(u))\phi(u)^{-2} \left[ \phi(u) \int_{\Omega} g(x, u) u \, dx - (2\sigma)^2 \vartheta(u) f_\varphi(u) f'_\varphi(u)(u) \right] \]

where we have set

\[ \vartheta(u) = \phi(u)^{-1} \int_{\Omega} (G(x, u) + c_2) \, dx. \]

Moreover, a direct computation yields:

\[ \tilde{f}_\varphi(u)(u) = (1 + T_1(u)) \int_{\Omega} \sum_{i,j=1}^{n} a_{ij}(x, u) D_i u D_j u \, dx + \]

\[ + \frac{1}{2} (1 + T_1(u)) \int_{\Omega} \sum_{i,j=1}^{n} D_s a_{ij}(x, u) D_i u D_j u \, dx + \]

\[ - (1 + T_2(u)) \int_{\Omega} g(x, u) u \, dx - (\psi(u) + T_1(u)) \int_{\Omega} \varphi u \, dx, \]

where \( T_1, T_2 : H^1_0(\Omega) \to \mathbb{R} \) are defined by setting:

\[ T_1(u) = \chi'(\vartheta(u))(2\sigma)^2 \vartheta(u)\phi(u)^{-2} f_\varphi(u) \int_{\Omega} \varphi u \, dx, \]

and

\[ T_2(u) = \chi'(\vartheta(u))\phi(u)^{-1} \int_{\Omega} \varphi u \, dx + T_1(u). \]

At this point, argue on the term \( \tilde{f}_\varphi(u) - \frac{1}{2(1+T_1(u))} \tilde{f}_\varphi(u)(u) \) as in Lemma 2.1.

3 The concrete Palais–Smale condition

We now introduce a variant of the classical Palais–Smale condition that is more suitable in our nonsmooth context.

**Definition 3.1** A sequence \( (u_h) \) in \( H^1_0(\Omega) \) is said to be a concrete Palais–Smale sequence at level \( c \in \mathbb{R} \) \( (\text{CPS})_c \)-sequence, in short) for the functional \( \tilde{f}_\varphi \), if \( \tilde{f}_\varphi(u_h) \to c \),

\[ \sum_{i,j=1}^{n} D_s a_{ij}(x, u_h) D_i u_h D_j u_h \in H^{-1}(\Omega). \]
eventually as $h \to \infty$ and

$$- \sum_{i,j=1}^{n} D_j(a_{ij}(x,u_h)D_iu_h) + \frac{1}{2} \sum_{i,j=1}^{n} D_ja_{ij}(x,u_h)D_iu_hD_ju_h - \bar{g}(x,u_h) \to 0$$

strongly in $H^{-1}(\Omega)$. We say that $\tilde{f}_\varphi$ satisfies the concrete Palais–Smale condition at level $c$ ((CPS)$_c$ condition), if every (CPS)$_c$–sequence for $\tilde{f}_\varphi$ admits a strongly convergent subsequence in $H^1_0(\Omega)$.

**Lemma 3.2** There exists $\bar{M} \in \mathbb{R}$ such that each (CPS)$_c$–sequence $(u_h)$ for $\tilde{f}_\varphi$ with $c \geq \bar{M}$ is bounded in $H^1_0(\Omega)$.

**Proof.** Let $K > 0$ be such that for large $h \in \mathbb{N}$ and any $\varrho > 0$, we have:

$$\varrho \|u_h\|_{1,2} + K \geq \tilde{f}_\varphi(u_h) - \varrho \tilde{f}_\varphi(u_h)(u_h).$$

If we choose $\gamma'$ and $\varepsilon$ as in the proof of Lemma 2.1, by inequality (7), arguing as in the proof of [13, Lemma 3.2], we have:

$$\varrho \|u_h\|_{1,2} + K \geq$$

$$\geq \left( \frac{1}{2} - \varrho(1 + T_1(u_h)) - \frac{\varrho\gamma'}{2}(1 + T_1(u_h)) \right) \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_h)D_iu_hD_ju_h \, dx +$$

$$+ \varrho(1 + T_2(u_h)) \int_\Omega g(x,u_h) \, u_h \, dx - \int_\Omega G(x,u_h) \, dx +$$

$$+ [\varrho(\psi(u_h) + T_1(u_h)) - \psi(u_h)] \int_\Omega \varphi \, u_h \, dx - \frac{\varrho}{2}(1 + T_1(u_h)) M_{R',\varepsilon} \geq$$

$$\geq \frac{\nu}{2} (1 - \varrho(2 + \gamma')(1 + T_1(u_h))) \|u_h\|_{1,2}^2 + (\varrho g(1 + T_2(u_h)) - 1) \int_\Omega G(x,u_h) \, dx$$

$$- [\varrho(1 + T_1(u_h)) + 1]\|\varphi\|_2 \|u_h\|_2 - \frac{\varrho}{2}(1 + T_1(u_h)) M_{R',\varepsilon}.$$ 

If we take $\bar{M}$ sufficiently large, we find $\delta > 0$, $\eta > 0$ and $\varrho \in ]\frac{1+n}{q}, \frac{1-\delta}{\gamma'+2}[$ with

$$(1 - \varrho(2 + \gamma')(1 + T_1(u_h))) > \delta, \quad (\varrho g(1 + T_2(u_h)) - 1) > \eta,$$

uniformly in $h \in \mathbb{N}$. Hence we obtain:

$$\varrho \|u_h\|_{1,2} + K \geq \frac{\nu \delta}{2} \|u_h\|_{1,2}^2 + b \eta \|u_h\|_q^q - c \|u_h\|_{1,2} - d_{R',\varepsilon},$$

which implies that the sequence $(u_h)$ is bounded in $H^1_0(\Omega)$.

The next result is one of the main tools to get our existence result.

**Theorem 3.3** $\tilde{f}_\varphi$ satisfies the (CPS)$_c$ condition at each level $c \geq \bar{M}$.
Proof. Let \((u_h)\) be a \((CPS)_c\)-sequence for \(\tilde{f}_\varphi\) with \(c \geq \tilde{M}\), where \(\tilde{M}\) is as in Lemma 3.2. Therefore \((u_h)\) is bounded in \(H^1_0(\Omega)\) and from [13, Lemma 3.3] we deduce that, up to subsequences, \((\tilde{g}(x, u_h))\) is strongly convergent in \(H^{-1}(\Omega)\). Then, by [7, Theorem 2.2.4], there exists a further subsequence \((u_{h_k})\) which strongly converges in \(H^1_0(\Omega)\).

4 Comparison of min–max values

In this section, we shall build two min–max classes for \(\tilde{f}_\varphi\) and then we shall compare the growths of the associated min–max values (see [15]). Let \((u_h)\) be the orthonormalized sequence of solutions to the problem:

\[
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
u u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

and set \(V_0 = \langle u_0 \rangle\) and \(V_{k+1} = V_k \oplus \mathbb{R}u_{k+1}\) for each \(k \geq 1\). Since each \(V_k\) is finite dimensional, one can find \(\beta_1, \beta_2, \beta_3 > 0\) such that:

\[
\tilde{f}_\varphi(u) \leq \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^q - \beta_3,
\]

for each \(u \in V_k\). In particular, for each \(k \in \mathbb{N}\) there exists \(R_k > 0\) such that:

\[
\|u\|_{1,2} \geq R_k \implies \tilde{f}_\varphi(u) \leq \tilde{f}_\varphi(0) \leq 0
\]

for all \(u \in V_k\).

**Definition 4.1** For each \(k \in \mathbb{N}\) set \(D_k = V_k \cap B(0, R_k)\),

\[
\Gamma_k = \left\{ \gamma \in C(D_k, H^1_0(\Omega)) : \gamma \text{ odd and } \gamma|_{B(0, R_k)} = Id \right\},
\]

and

\[
b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \tilde{f}_\varphi(\gamma(u)).
\]

**Lemma 4.2** There exist \(\beta > 0\) and \(k_0 \in \mathbb{N}\) with \(b_k \geq \beta k^{\frac{2(q+1)}{n(q-1)}}\) for \(k \geq k_0\).

**Proof.** Since there exist \(\beta_1, \beta_2 > 0\) such that

\[
\tilde{f}_\varphi(u) \geq \frac{\nu}{2} \int_\Omega |Du|^2 dx - \beta_1 \|u\|_{\sigma+1}^{\sigma+1} - \beta_2,
\]

\[
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\]

\[
\tilde{f}_\varphi(u) \leq \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^q - \beta_3,
\]

for each \(u \in V_k\). In particular, for each \(k \in \mathbb{N}\) there exists \(R_k > 0\) such that:

\[
\|u\|_{1,2} \geq R_k \implies \tilde{f}_\varphi(u) \leq \tilde{f}_\varphi(0) \leq 0
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\]

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**Proof.** Since there exist \(\beta_1, \beta_2 > 0\) such that

\[
\tilde{f}_\varphi(u) \geq \frac{\nu}{2} \int_\Omega |Du|^2 dx - \beta_1 \|u\|_{\sigma+1}^{\sigma+1} - \beta_2,
\]
it suffices to follow the proof of [19, Theorem 1].

**Definition 4.3** We denote by $U_k$ the set of $\xi = tu_{k+1} + w$ such that:

$$0 \leq t \leq R_{k+1}, \ w \in B(0, R_{k+1}) \cap V_k, \ \|\xi\|_{1,2} \leq R_{k+1}.$$ 

We denote by $\Lambda_k$ the set of $\lambda \in C(U_k, H^1_0(\Omega))$ such that:

$$\lambda|_{D_k} \in \Gamma_{k+1}, \ \lambda|_{\partial B(0,R_{k+1}) \cup (B(0,R_{k+1}) \setminus B(0,R_k)) \cap V_k} = Id$$

and we set:

$$c_k = \inf_{\lambda \in \Lambda_k} \max_{u \in U_k} \overline{f}_\varphi(\lambda(u)).$$

The next is our main existence tool.

**Lemma 4.4** Assume that $c_k > b_k \geq \widetilde{M}$. If $\delta \in ]0, c_k - b_k[$ and

$$\Lambda_k(\delta) = \left\{ \lambda \in \Lambda_k : \overline{f}_\varphi(\lambda(u)) \leq b_k + \delta \text{ for } u \in D_k \right\}$$

set

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \max_{u \in U_k} \overline{f}_\varphi(\lambda(u)).$$

Then $c_k(\delta)$ is a critical value for $\overline{f}_\varphi$.

**Proof.** See [13, Lemma 5.5]. Of course, differently from the proof of [15, Lemma 1.57], in this nonsmooth framework, we shall apply [7, Theorem 1.1.13] instead of the classical Deformation Lemma (see Lemma 1.60 of [15]).

**Lemma 4.5** Assume that $c_k = b_k$ for all $k \geq k_1$. Then, there exist $\gamma > 0$ such that $b_k \leq \gamma k^{\frac{2}{4-s}}$ for each $k \geq k_1$.

**Proof.** See [13, Lemma 5.6].

We finally come to the proof of our main result.

**Proof of Theorem 1.1.** The restriction on $\sigma$ implies that $q/(q-1) < (2(\sigma + 1))/(n(\sigma - 1))$. Therefore, combining Lemma 4.2 and Lemma 4.5 we deduce that $c_k > b_k$, so that we may apply Lemma 4.4 and obtain that $(c_k(\delta))$ is a sequence of critical values for $\overline{f}_\varphi$. Finally, if $M$ is larger than $\max\{\widetilde{M}, \overline{M}\}$,
by Theorem 2.3 we conclude that $f_\phi$ has a diverging sequence of critical values. □

Proof of Theorem 1.2. It is a variant of the proof of Theorem 1.1. It suffices to slightly modify the estimates in several of the Lemmas. □

References


