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Multiple critical points for perturbed symmetric functionals associated with quasilinear elliptic problems

M. Squassina

Dipartimento di Matematica e Fisica, Università Cattolica del Sacro Cuore, Via dei Musei 41, I 25121 Brescia, Italy

Abstract

By means of nonsmooth critical point theory, we prove existence of infinitely many weak solutions for a class of perturbed symmetric quasilinear elliptic equations.

Key words: Nonsmooth critical point theory, perturbation of symmetric functionals, quasilinear elliptic equations.
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1 Introduction

The main goal of this paper is to extend to the quasilinear case the existence results known since 1980 for the semilinear scalar problem

$$\begin{cases} -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu) = g(x,u) + \varphi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with g is superlinear and odd in u, $\varphi \in L^2(\Omega)$ and Ω is a bounded domain in $\mathbb{R}^n$.

This problem has been deeply studied in [2], [10], [15], [17] and [19] by means of the variational techniques developed by Bahri, Berestycki, Rabinowitz and Struwe in the early eighties. We remark that, around 1990, Bahri and P. L. Lions improved in [3] and [4] the previous results via a technique based on

[1] Present address: Dipartimento di Matematica, Università degli studi di Milano, Via C. Saldini 50, I 20133 Milano, Italy. e-mail: squassin@mat.unimi.it

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Morse theory. See also [20] for further improvements by means of a completely different method devised by P. Bolle.

On the other hand, since 1994, several efforts have been devoted to study existence for quasilinear problems of the type

$$\begin{cases} -\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We refer the reader to [5], [6], [7] and [18] for the study of multiplicity of solutions of problem (1) and furthermore to [1], [14] and [16] for an even more general framework.

The functional $f_0 : H_0^1(\Omega) \to \mathbb{R}$ associated with (1) is given by

$$f_0(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega G(x,u)\,dx\,.$$

We stress that f_0 is not even locally Lipschitz unless the a_{ij}'s do not depend on u.

Consequently, techniques of nonsmooth critical point theory have to be applied. We refer the reader to [7], [8], [9], [11] and [12] for the abstract theory that we shall need in the following.

It seems now natural to ask whether also in a quasilinear setting the multiplicity of solutions is stable under large L^2–perturbations.

In this paper we want to investigate the effects of destroying the symmetry of (1) and show that for each $\varphi \in L^2(\Omega)$ the perturbed equation

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) + \varphi \quad \text{in } \Omega \tag{2}$$

still has infinitely many weak solutions.

Therefore, we shall work on the functional

$$f_\varphi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega G(x,u)\,dx - \int_\Omega \varphi u\,dx\,.$$

In the next, Ω will denote an open and bounded subset of $\mathbb{R}^n$ with $n \geqslant 3$. Moreover, we shall consider the following assumptions :

$(\mathscr{H}_1)$ each $a_{ij}(x,s)$ is measurable in x for each $s \in \mathbb{R}$ and of class C^1 in s for a.e. $x \in \Omega$ with $a_{ij}(x,s) = a_{ji}(x,s)$, $a_{ij}(\cdot,\cdot) \in L^\infty(\Omega \times \mathbb{R})$ and $D_s a_{ij}(\cdot,\cdot) \in L^\infty(\Omega \times \mathbb{R})$. Moreover, there exist $\nu > 0$ and $R > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j \geqslant \nu|\xi|^2$$

$$|s| \geqslant R \Longrightarrow \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \geqslant 0\,, \tag{3}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$;

$(\mathscr{H}_2)$ $G(x,s)$ is measurable in x for all $s \in \mathbb{R}$, of class C^1 in s a.e. in Ω with $G(x,0) = 0$ and $g(x,s) = D_sG(x,s)$. Moreover, there exist $q > 2$ and $R' > 0$ with

$$|s| \geqslant R' \Longrightarrow 0 < qG(x,s) \leqslant g(x,s)s\,, \tag{4}$$

for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$;

$(\mathscr{H}_3)$ there exists $\gamma \in]0, q-2[$ such that :

$$|s| \geqslant R' \Longrightarrow \sum_{i,j=1}^{n} sD_s a_{ij}(x,s)\xi_i\xi_j \leqslant \gamma \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j\,, \tag{5}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$.

Under the previous assumptions, the following is our main result.

Theorem 1.1 *Assume that there exists* $\sigma \in \left]1, \frac{qn+2(q-1)}{qn-2(q-1)}\right[$ *such that*

$$|g(x,s)| \leqslant a + b|s|^\sigma$$

with $a, b \in \mathbb{R}$ *and that for a.e.* $x \in \Omega$ *and for each* $s \in \mathbb{R}$

$$a_{ij}(x,-s) = a_{ij}(x,s)\,, \qquad g(x,-s) = -g(x,s)\,.$$

Then there exists a sequence (u_h) *of weak solutions to* (2) *with* $f_\varphi(u_h) \to +\infty$.

Since for each $q > 2$ and $n \geqslant 3$ we have

$$\frac{qn+(q-1)(n+2)}{qn+(q-1)(n-2)} < \frac{qn+2(q-1)}{qn-2(q-1)},$$

this result relaxes Theorem 1.1 of [13] which was proven for

$$\sigma \in \left]1, \frac{qn+(q-1)(n+2)}{qn+(q-1)(n-2)}\right[.$$

This because we used in Lemma 4.2 the sharp estimate from below on the growth of the critical values shown by K. Tanaka in 1989 via Morse theory [19] which improves the direct estimate of P. H. Rabinowitz of 1981 based on the Gagliardo–Niremberg inequality.

We point out that we assumed (3) and (5) only for large values of $|s|$, while in [13], dealing with systems, we requested conditions (3) and (5) to hold for each $s \in \mathbb{R}$.

In the next result we allow a more general class of perturbations.

Theorem 1.2 *Assume that there exists $\sigma \in]1, 2^*[$ and $\sigma' \in [0, q-1[$ with*

$$|g(x,s)| \leqslant a + b|s|^{\sigma}, \qquad |\varphi(x,s)| \leqslant c + d|s|^{\sigma'},$$

where $\varphi \in C(\Omega \times \mathbb{R})$ and $\frac{2(\sigma+1)}{n(\sigma-1)} > \frac{q}{q-\sigma'-1}$. Assume further that :

$$a_{ij}(x,-s) = a_{ij}(x,s)\,, \qquad g(x,-s) = -g(x,s)\,.$$

Then there exists a sequence (u_h) of weak solutions of the problem

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u)D_iu) + \tfrac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u)D_iuD_ju = g(x,u) + \varphi(x,u) \text{ in } \Omega$$

with $u = 0$ on $\partial\Omega$, such that $f_{\Phi}(u_h) \to +\infty$ where :

$$f_{\Phi}(u) = \tfrac{1}{2}\int_{\Omega}\sum_{i,j=1}^{n} a_{ij}(x,u)D_iuD_ju\,dx - \int_{\Omega} G(x,u)\,dx - \int_{\Omega}\Phi(x,u)\,dx\,,$$

and $D_s\Phi(x,s) = \varphi(x,s)$ for each $x \in \Omega$ and all $s \in \mathbb{R}$.

These theorems extend the results of [2], [10], [15] and [17] to the quasilinear case.

2 Perturbation of even functionals

If $\varphi \not\equiv 0$, clearly f_φ is not even. Note that by (4) we find $c_1, c_2, c_3 > 0$ such that :

$$\frac{1}{q}(g(x,s)s + c_1) \geqslant G(x,s) + c_2 \geqslant c_3|s|^q\,, \tag{6}$$

for a.e. $x \in \Omega$ and each $s \in \mathbb{R}$.

Lemma 2.1 *Assume that u is a weak solution to (2). Then it results*

$$\int_\Omega (G(x,u) + c_2)\; dx \leqslant \sigma \left(f_\varphi^2(u) + 1\right)^{\frac{1}{2}},$$

for some $\sigma > 0$ depending on $\|\varphi\|_2$.

Proof. Let us set $C = \|D_s a_{ij}\|_{L^\infty(\Omega\times\mathbb{R})}$. If we choose $\gamma' \in]\gamma, q-2[$ and $\varepsilon > 0$ in such a way that

$$\frac{nCR'\varepsilon}{\nu} \leqslant \gamma' - \gamma,$$

using [7, Theorem 2.2.9] and working as in the proof of [7, Lemma 2.3.2], we get

$$\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u\, u\, dx \leqslant$$
$$\leqslant \gamma' \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u\, dx + M_{R',\varepsilon}. \tag{7}$$

Therefore, we deduce that :

$$\begin{aligned}
&f_\varphi(u) = f_\varphi(u) - \frac{1}{2} f'_\varphi(u)(u) = \\
&= \int_\Omega \left[\frac{1}{2} g(x,u)\, u - G(x,u) - \frac{1}{2}\varphi\, u\right] dx - \frac{1}{4}\int_\Omega \sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u\, u\, dx \geqslant \\
&\geqslant \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega (g(x,u)\, u + c_1)\, dx - \frac{\|\varphi\|_2 \|u\|_2}{2} + \\
&\quad - \frac{\gamma'}{4} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u\, dx - c_4 \geqslant \\
&\geqslant \left(\frac{q}{2} - 1 - \frac{\gamma'}{2}\right) \int_\Omega (G(x,u) + c_2)\; dx - \frac{\gamma'}{2} f_\varphi(u) - \delta \|u\|_q^q - \beta(\delta) \|\varphi\|_2^{q'} - c_5
\end{aligned}$$

with $\delta \to 0$ and $\beta(\delta) \to +\infty$. Choosing $\delta > 0$ small enough, by (6) we have :

$$\sigma_\delta f_\varphi(u) \geqslant \int_\Omega (G(x,u) + c_2)\, dx - c_6,$$

where $\sigma_\delta = \frac{2+\gamma'}{q-2-\gamma'-2\delta}$. The assertion follows as in [15, Lemma 1.8].

Let us now define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi = 1$ for $s \leqslant 1$, $\chi = 0$ for $s \geqslant 2$ and $-2 < \chi' < 0$ when $1 < s < 2$, and let us set :

$$\phi(u) = 2\sigma \left(f_\varphi^2(u) + 1\right)^{\frac{1}{2}}, \qquad \psi(u) = \chi\left(\phi(u)^{-1} \int_\Omega (G(x,u) + c_2)\, dx\right)$$

for each $u \in H_0^1(\Omega)$. Finally, we define the modified functional by setting :

$$\widetilde{f}_\varphi(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u\, dx - \int_\Omega G(x,u)\, dx - \psi(u) \int_\Omega \varphi\, u\, dx \tag{8}$$

The Euler's equation associated with (8) is given by :

$$\begin{cases} -\sum\limits_{i,j=1}^n D_j(a_{ij}(x,u) D_i u) + \frac{1}{2} \sum\limits_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u = \widetilde{g}(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where we have set :

$$\widetilde{g}(x,u) = g(x,u) + \psi(u)\varphi + \psi'(u) \int_\Omega \varphi\, u\, dx\,.$$

We remark that by Lemma 2.1, if u solves (2), then $\psi(u) = 1$ and $\widetilde{f}_\varphi(u) = f_\varphi(u)$. In the next result, we measure the defect of symmetry of $\widetilde{f}_\varphi$.

Lemma 2.2 *There exists $\beta > 0$ depending on $\|\varphi\|_2$ such that*

$$\left|\widetilde{f}_\varphi(u) - \widetilde{f}_\varphi(-u)\right| \leqslant \beta \left\{ |\widetilde{f}_\varphi(u)|^{\frac{1}{q}} + 1 \right\}$$

for each $u \in H_0^1(\Omega)$.

Proof. See [13, Lemma 2.2].

Theorem 2.3 *There exists $\widehat{M} > 0$ such that if u is a weak solution of* (9) *such that $\widetilde{f}_\varphi(u) \geqslant \widehat{M}$ then u is a weak solution to* (2) *and $\widetilde{f}_\varphi(u) = f_\varphi(u)$.*

Proof. For the complete proof, see [13, Theorem 2.3]. Let us give a brief sketch. Standard computations yield :

$$\psi'(u)(u) = \chi'(\vartheta(u))\phi(u)^{-2}\left[\phi(u)\int_\Omega g(x,u)u\,dx - (2\sigma)^2\vartheta(u)f_\varphi(u)f'_\varphi(u)(u)\right]$$

where we have set

$$\vartheta(u) = \phi(u)^{-1}\int_\Omega (G(x,u)+c_2)\;dx\,.$$

Moreover, a direct computation yields :

$$\begin{aligned}\widetilde{f}'_\varphi(u)(u) =&(1+T_1(u))\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u)D_iuD_ju\,dx + \\ &+\frac{1}{2}(1+T_1(u))\int_\Omega \sum_{i,j=1}^n D_sa_{ij}(x,u)D_iuD_ju\,u\,dx + \\ &-(1+T_2(u))\int_\Omega g(x,u)\,u\,dx - (\psi(u)+T_1(u))\int_\Omega \varphi\,u\,dx\,,\end{aligned}$$

where $T_1, T_2 : H^1_0(\Omega) \to \mathbb{R}$ are defined by setting :

$$T_1(u) = \chi'(\vartheta(u))(2\sigma)^2\vartheta(u)\phi(u)^{-2}f_\varphi(u)\int_\Omega \varphi\,u\,dx\,,$$

and

$$T_2(u) = \chi'(\vartheta(u))\phi(u)^{-1}\int_\Omega \varphi\,u\,dx + T_1(u).$$

At this point, argue on the term $\widetilde{f}_\varphi(u) - \frac{1}{2(1+T_1(u))}\widetilde{f}'_\varphi(u)(u)$ as in Lemma 2.1.

3 The concrete Palais–Smale condition

We now introduce a variant of the classical Palais–Smale condition that is more suitable in our nonsmooth context.

Definition 3.1 *A sequence (u_h) in $H^1_0(\Omega)$ is said to be a concrete Palais–Smale sequence at level $c \in \mathbb{R}$ ($(CPS)_c$–*sequence, *in short) for the functional $\widetilde{f}_\varphi$, if $\widetilde{f}_\varphi(u_h) \to c$,*

$$\sum_{i,j=1}^n D_sa_{ij}(x,u_h)D_iu_hD_ju_h \in H^{-1}(\Omega)$$

eventually as $h \to \infty$ *and*

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x,u_h)D_i u_h) + \tfrac{1}{2}\sum_{i,j=1}^{n} D_s a_{ij}(x,u_h)D_i u_h D_j u_h - \widetilde{g}(x,u_h) \to 0$$

strongly in $H^{-1}(\Omega)$ *. We say that* $\widetilde{f}_\varphi$ *satisfies the concrete Palais–Smale condition at level c* (($(CPS)_c$ *condition), if every* $(CPS)_c$*−sequence for* $\widetilde{f}_\varphi$ *admits a strongly convergent subsequence in* $H_0^1(\Omega)$.

Lemma 3.2 *There exists* $\widetilde{M} \in \mathbb{R}$ *such that each* $(CPS)_c$*−sequence* (u_h) *for* $\widetilde{f}_\varphi$ *with* $c \geqslant \widetilde{M}$ *is bounded in* $H_0^1(\Omega)$.

Proof. Let $K > 0$ be such that for large $h \in \mathbb{N}$ and any $\varrho > 0$, we have :

$$\varrho\|u_h\|_{1,2} + K \geqslant \widetilde{f}_\varphi(u_h) - \varrho \widetilde{f}'_\varphi(u_h)(u_h)\,.$$

If we choose γ' and ε as in the proof of Lemma 2.1, by inequality (7), arguing as in the proof of [13, Lemma 3.2], we have :

$$\begin{aligned}
&\varrho\|u_h\|_{1,2} + K \geqslant \\
&\geqslant \left(\frac{1}{2} - \varrho(1+T_1(u_h)) - \frac{\varrho\gamma'}{2}(1+T_1(u_h))\right)\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x,u_h)D_i u_h D_j u_h\,dx + \\
&+\varrho(1+T_2(u_h))\int_\Omega g(x,u_h)\,u_h\,dx - \int_\Omega G(x,u_h)\,dx + \\
&+[\varrho(\psi(u_h)+T_1(u_h)) - \psi(u_h)]\int_\Omega \varphi\,u_h\,dx - \frac{\varrho}{2}(1+T_1(u_h))M_{R',\varepsilon} \geqslant \\
&\geqslant \frac{\nu}{2}\left(1-\varrho\,(2+\gamma')\,(1+T_1(u_h))\right)\|u_h\|_{1,2}^2 + (q\varrho(1+T_2(u_h))-1)\int_\Omega G(x,u_h)\,dx \\
&-[\varrho(1+T_1(u_h))+1]\|\varphi\|_2\|u_h\|_2 - \frac{\varrho}{2}(1+T_1(u_h))M_{R',\varepsilon}\,.
\end{aligned}$$

If we take $\widetilde{M}$ sufficiently large, we find $\delta > 0$, $\eta > 0$ and $\varrho \in \left]\frac{1+\eta}{q}, \frac{1-\delta}{\gamma'+2}\right[$ with

$$(1-\varrho\,(2+\gamma')\,(1+T_1(u_h))) > \delta, \qquad (q\varrho(1+T_2(u_h))-1) > \eta\,,$$

uniformly in $h \in \mathbb{N}$. Hence we obtain :

$$\varrho\|u_h\|_{1,2} + K \geqslant \frac{\nu\delta}{2}\|u_h\|_{1,2}^2 + b\eta\|u_h\|_q^q - c\|u_h\|_{1,2} - d_{R',\varepsilon},$$

which implies that the sequence (u_h) is bounded in $H_0^1(\Omega)$.

The next result is one of the main tools to get our existence result.

Theorem 3.3 $\widetilde{f}_\varphi$ *satisfies the* $(CPS)_c$ *condition at each level* $c \geqslant \widetilde{M}$.

Proof. Let (u_h) be a $(CPS)_c$–sequence for $\widetilde{f}_\varphi$ with $c \geqslant \widetilde{M}$, where $\widetilde{M}$ is as in Lemma 3.2. Therefore (u_h) is bounded in $H_0^1(\Omega)$ and from [13, Lemma 3.3] we deduce that, up to subsequences, $(\widetilde{g}(x, u_h))$ is strongly convergent in $H^{-1}(\Omega)$. Then, by [7, Theorem 2.2.4], there exists a further subsequence (u_{h_k}) which strongly converges in $H_0^1(\Omega)$.

4 Comparison of min–max values

In this section, we shall build two min–max classes for $\widetilde{f}_\varphi$ and then we shall compare the growths of the associated min–max values (see [15]). Let (u_h) be the orthonormalized sequence of solutions to the problem :

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and set $V_0 = \langle u_0 \rangle$ and $V_{k+1} = V_k \oplus \mathbb{R} u_{k+1}$ for each $k \geqslant 1$. Since each V_k is finite dimentional, one can find $\beta_1, \beta_2, \beta_3 > 0$ such that :

$$\widetilde{f}_\varphi(u) \leqslant \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^q - \beta_3\,,$$

for each $u \in V_k$. In particular, for each $k \in \mathbb{N}$ there exists $R_k > 0$ such that :

$$\|u\|_{1,2} \geqslant R_k \Longrightarrow \widetilde{f}_\varphi(u) \leqslant \widetilde{f}_\varphi(0) \leqslant 0$$

for all $u \in V_k$.

Definition 4.1 *For each $k \in \mathbb{N}$ set $D_k = V_k \cap B(0, R_k)$,*

$$\Gamma_k = \left\{ \gamma \in C(D_k, H_0^1(\Omega)) : \ \gamma \text{ odd and } \gamma_{|\partial B(0,R_k)} = Id \right\},$$

and

$$b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \widetilde{f}_\varphi(\gamma(u))\,.$$

Lemma 4.2 *There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ with $b_k \geqslant \beta k^{\frac{2(\sigma+1)}{n(\sigma-1)}}$ for $k \geqslant k_0$.*

Proof. Since there exist $\beta_1, \beta_2 > 0$ such that

$$\widetilde{f}_\varphi(u) \geqslant \frac{\nu}{2} \int_\Omega |Du|^2 \, dx - \beta_1 \|u\|_{\sigma+1}^{\sigma+1} - \beta_2\,,$$

it suffices to follow the proof of [19, Theorem 1].

Definition 4.3 *We denote by U_k the set of $\xi = tu_{k+1} + w$ such that :*

$$0 \leqslant t \leqslant R_{k+1}, \ \ w \in B(0, R_{k+1}) \cap V_k, \ \ \|\xi\|_{1,2} \leqslant R_{k+1}\,.$$

We denote by Λ_k the set of $\lambda \in C(U_k, H_0^1(\Omega))$ such that :

$$\lambda_{|_{D_k}} \in \Gamma_{k+1}, \ \ \lambda|_{\partial B(0,R_{k+1}) \cup ((B(0,R_{k+1}) \setminus B(0,R_k)) \cap V_k)} = Id$$

and we set :

$$c_k = \inf_{\lambda \in \Lambda_k} \max_{u \in U_k} \widetilde{f}_\varphi(\lambda(u))\,.$$

The next is our main existence tool.

Lemma 4.4 *Assume that $c_k > b_k \geqslant \widetilde{M}$. If $\delta \in]0, c_k - b_k[$ and*

$$\Lambda_k(\delta) = \left\{ \lambda \in \Lambda_k : \ \ \widetilde{f}_\varphi(\lambda(u)) \leqslant b_k + \delta \ \text{ for } \ u \in D_k \right\}$$

set

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \max_{u \in U_k} \widetilde{f}_\varphi(\lambda(u))\,.$$

Then $c_k(\delta)$ is a critical value for $\widetilde{f}_\varphi$.

Proof. See [13, Lemma 5.5]. Of course, differently from the proof of [15, Lemma 1.57], in this nonsmooth framework, we shall apply [7, Theorem 1.1.13] instead of the classical Deformation Lemma (see Lemma 1.60 of [15]).

Lemma 4.5 *Assume that $c_k = b_k$ for all $k \geqslant k_1$. Then, there exist $\gamma > 0$ such that $b_k \leqslant \gamma k^{\frac{q}{q-1}}$ for each $k \geqslant k_1$.*

Proof. See [13, Lemma 5.6].

We finally come to the proof of our main result.

Proof of Theorem 1.1. The restriction on σ implies that $q/(q-1) < (2(\sigma + 1))/(n(\sigma - 1))$. Therefore, combining Lemma 4.2 and Lemma 4.5 we deduce that $c_k > b_k$, so that we may apply Lemma 4.4 and obtain that $(c_k(\delta))$ is a sequence of critical values for $\widetilde{f}_\varphi$. Finally, if M is larger than $\max\{\widetilde{M}, \widehat{M}\}$, by Theorem 2.3 we conclude that f_φ has a diverging sequence of critical values. □

Proof of Theorem 1.2. It is a variant of the proof of Theorem 1.1. It suffices to slightly modify the estimates in several of the Lemmas. □

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