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# Multiple solutions for quasilinear elliptic problems in $\mathbb{R}^2$ with exponential growth

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**Abstract.** By combining techniques of nonsmooth critical point theory with a sharp estimate of Trudinger–Moser type, we prove the existence of an infinite number of solutions for a class of perturbed symmetric elliptic problems at exponential growth in $\mathbb{R}^2$ covering the full range of subcriticality allowed.

## 1. Introduction

In 1994 K. Sugimura proved that, given an open bounded domain $\Omega$ of $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, for each $\varphi \in L^2(\Omega)$ the semilinear elliptic problem

$$\begin{cases} -\Delta u = g(x,u) + \varphi & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

admits an unbounded sequence of solutions $(u_h) \subset H_0^1(\Omega)$ provided that $g(x,u)$ is an odd (in $u$) superlinear nonlinearity with exponential growth such that

$$A_1 e^{|s|^{p_1}} - B_1 \leqslant \int_0^s g(x,\tau)\, d\tau \leqslant A_2 e^{|s|^{p_2}} - B_2 \quad 0 < p_1 \leqslant p_2 < \frac{1}{2},$$

a.e. in $\Omega$ and for each $s \in \mathbb{R}$, where $A_1, A_2 > 0$ and $B_1, B_2 \geqslant 0$ (see [22]).

The main goal of this paper is improving Sugimura's result and at the same time extending these type of achievements to the case of quasilinear elliptic equations. For a planar domain $\Omega$, the analogue of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ in dimensions greater than 3 is the Orlicz space embedding

$$\forall s \geqslant 1: \quad H_0^1(\Omega) \ni u \mapsto e^{u^2} \in L^s(\Omega)$$

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for which the Trudinger–Moser inequality holds: there exists $C_{TM} > 0$ with

$$\forall u \in H_0^1(\Omega) : \|u\|_{1,2} \leqslant 1 \Longrightarrow \int_\Omega e^{4\pi u^2}\, dx \leqslant C_{TM}\mathscr{L}^2(\Omega), \tag{2}$$

where $\mathscr{L}^2$ denotes the usual Lebesgue measure in $\mathbb{R}^2$ and $\|\cdot\|_{1,2}$ is the standard norm in $H_0^1(\Omega)$. In view of a sharp inequality like (2) (see Theorem 5), we shall obtain a multiplicity result for the exponential nonlinearity

$$\forall s \in \mathbb{R} : \quad g(s) = |s|^{p-2} s e^{|s|^p},$$

all over the subcritical range $1 < p < 2$.

Let us now briefly recall the historic background of the problem of broken symmetry for elliptic equations. If $\Omega$ is a bounded domain of $\mathbb{R}^n$ with $n \geqslant 2$, the multiplicity of solutions for semilinear elliptic problems of the type

$$\begin{cases} -\Delta u = g(x,u) + \varphi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

with $g$ superlinear, odd in $u$ and for $a, b > 0$

$$|g(x,s)| \leqslant a + b|s|^p, \quad 1 < p < \sigma \leqslant 2^* - 1 \quad \text{if } n \geqslant 3,$$
$$1 < p < +\infty \quad \text{if } n = 1, 2,$$

has been investigated by the variational techniques developed by Bahri, Berestycki, Rabinowitz and Struwe in the early eighties [3,11,16,20,23]. Later on, around 1990, Bahri and Lions improved the previous results via a technique based on Morse theory (see [4,5]).

Very recently further improvements have been achieved by a completely new method devised by P. Bolle (see [6]). When $n = 2$, the result of Bahri and Lions [4] is optimal for the power case $g(x,s) = |s|^{p-1}s$, namely the multiplicity appears for all $p > 1$. However, when $n \geqslant 3$, it remains open the problem of whether (3) has an infinite number of solutions for all $\sigma$ all the way up to the exponent $2^* - 1$.

Since 1994, several works have been devoted to the study of quasilinear elliptic equations of the type:

$$-\sum_{i,j=1}^n D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^n D_s a_{ij}(x,u) D_i u D_j u = g(x,u) \quad \text{in } \Omega, \tag{4}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^n$ with $n \geqslant 3$. We refer the reader to [7,8] for the study of multiplicity of solutions of this problem and furthermore to [2] and [18] for an even more general framework. The functional $f_0 : H_0^1(\Omega) \to \mathbb{R}$ associated with (4) is given by

$$f_0(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u\, dx - \int_\Omega G(x,u)\, dx,$$

where $D_s G(x, s) = g(x, s)$. As pointed out in [8], this functional fails to be smooth ($C^1$) for $n \geqslant 3$. On the other hand, also in the case $n = 2$, being

$$\forall s < +\infty : H_0^1(\Omega) \hookrightarrow L^s(\Omega) \quad \text{but} \quad H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega),$$

it may happen that

$$\sum_{i,j=1}^{2} D_s a_{ij}(x, u) D_i u D_j u \notin H^{-1}(\Omega),$$

even if $D_s a_{ij} \in L^\infty$, so that in general $f_0$ is continuous but fails to be locally Lipschitzian.

Consequently, techniques of nonsmooth critical point theory have to be emploied and the methods of [6] cannot be used since the functional is requested to be of class $C^2$. We refer the reader to [8–10, 12] and [14] for the abstract framework that we shall need in the following.

It seemed natural to ask whether also in the quasilinear setting the multiplicity of solutions persists under perturbations. A partial answer to this question has been given in [15] and [19] where it was proved that for a suitable $q > 2$, if

$$|g(x, s)| \leqslant a + b|s|^p, \quad 1 < p < \frac{qn + 2(q-1)}{qn - 2(q-1)},$$

with $a, b > 0$ and for each $i, j = 1, \ldots, n$

$$a_{ij}(x, -s) = a_{ij}(x, s), \quad g(x, -s) = -g(x, s)$$

a.e. in $\Omega$ and for each $s \in \mathbb{R}$, then for each $\varphi \in L^2(\Omega)$ the problem

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x, u) D_i u) + \frac{1}{2} \sum_{i,j=1}^{n} D_s a_{ij}(x, u) D_i u D_j u = g(x, u) + \varphi \tag{5}$$

with $u = 0$ on $\partial\Omega$, has an unbounded sequence $(u_h) \subset H_0^1(\Omega)$ of solutions.

A natural question is now whether the multiplicity of solutions appears for the perturbed equation (5) when $g$ possesses an exponential growth all along the subcritical range $1 < p < 2$. We are ready to give an answer to this question by stating the main result of the paper. In the next, $\Omega$ will denote a smooth bounded domain of $\mathbb{R}^2$. Moreover, we assume that:

($\mathscr{H}$) each $a_{ij}(x, s)$ is measurable in $x$ for each $s \in \mathbb{R}$ and of class $C^1$ in $s$ for a.e. $x \in \Omega$ with $a_{ij} = a_{ji}$, $a_{ij} \in L^\infty(\Omega \times \mathbb{R})$ and $D_s a_{ij} \in L^\infty(\Omega \times \mathbb{R})$. Moreover, there exist $\nu > 0$ and $R > 0$ such that:

$$\sum_{i,j=1}^{2} a_{ij}(x,s)\xi_i\xi_j \geqslant \nu|\xi|^2,$$
$$|s| \geqslant R \Longrightarrow \sum_{i,j=1}^{2} sD_s a_{ij}(x,s)\xi_i\xi_j \geqslant 0, \tag{6}$$

a.e. in $\Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^2$.

We point out that assumption (6) is well known in the current literature both for existence and regularity theory (see e.g. [2,7,8,15,18,19]).

Let $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous map and let $\sigma \geqslant 0$ be such that

$$|\varphi(x,s)| \leqslant a + b|s|^\sigma \quad (a,b > 0)$$

a.e. in $\Omega$ and for each $s \in \mathbb{R}$ and define $f_\varphi : H_0^1(\Omega) \to \mathbb{R}$ by setting

$$f_\varphi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u)D_i u D_j u\, dx - \int_\Omega \left(e^{|u|^p} - 1\right) dx - \int_\Omega \Phi(x,u)\, dx$$

with $D_s\Phi(x,s) = \varphi(x,s)$ for each $x \in \Omega$ and all $s \in \mathbb{R}$.

Under the preceding assumptions, the following is the main result.

**Theorem 1.** *Let $1 < p < 2$ and assume that*

$$a_{ij}(x,-s) = a_{ij}(x,s) \quad (i,j = 1,2)$$

*a.e. in $\Omega$ and for each $s \in \mathbb{R}$. Then the problem*

$$-\sum_{i,j=1}^{2} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{2} D_s a_{ij}(x,u)D_i u D_j u = p|u|^{p-2}ue^{|u|^p} + \varphi(x,u) \tag{7}$$

*with $u = 0$ on $\partial\Omega$, has a sequence $(u_h) \subset H_0^1(\Omega)$ of solutions such that*

$$\lim_h f_\varphi(u_h) = +\infty.$$

In particular, our result removes any upper bound in the subcritical growth completely. It has to be remarked that Theorem 1 is new also in the case $D_s a_{ij}(x,s) = 0$ a.e. in $\Omega$ and for each $s \in \mathbb{R}$ (semilinear case).

In the critical case $p = 2$, Adimurthi has conjectured in [1] that the problem

$$\begin{cases} -\Delta u = ue^{u^2} + \varphi & \text{in } B(0,1) \\ \quad u = 0 & \text{on } \partial B(0,1) \end{cases} \tag{8}$$

admits at most one positive solution $u \in H_0^1(B(0,1))$, where $B(0,1)$ is the unit ball in $\mathbb{R}^2$. On the other hand, this uniqueness result seems to be out of reach, so far.

The plan of the paper is as follows: in Sect. 2 we briefly recall some basic notions from the theory of Orlicz spaces. In Sect. 3, we recollect some definitions and results from nonsmooth critical point theory. In Sect. 4 we show how the functional associated with our problem satisfies a variant of the classical Palais–Smale condition. In Sects. 5 and 6 we obtain the key estimate from below (Lemma 3) and the estimate from above for the critical values associated with the minimax classes introduced by Rabinowitz in [16]. Finally, in Sect. 7, we end up the proof of Theorem 1. We point out that, for the sake of simplicity, we shall prove our result when $\varphi \in L^2(\Omega)$. The general case can be covered by slightly modifying several of the lemmas (see [16]).

## 2. Recalls from the theory of Orlicz spaces

Let us briefly recall some basic notions about Orlicz spaces that will be required later. For further details, we refer the interested reader to [17].

**Definition 1.** *Let $(\Omega, \Sigma, \mu)$ be an abstract measure space, where $\Omega$ is some point set, $\Sigma$ is a $\sigma$-algebra of its subsets on which a $\sigma$-additive function $\mu : \Sigma \to \mathbb{R}_+^*$ is given and $\mu$ has the finite subset property. Then, if $\Phi : \mathbb{R} \to \mathbb{R}_+^*$ is a Young function, we define*

$$\mathscr{O}_\mu^\Phi = \left\{ u : \Omega \to \mathbb{R}^* \text{ measurable with } \alpha u \in \mathscr{J}_\mu^\Phi \text{ for some } \alpha > 0 \right\},$$

*where*

$$\mathscr{J}_\mu^\Phi = \left\{ u : \Omega \to \mathbb{R}^* \text{ measurable for } \Sigma : \int_\Omega \Phi\left(|u|\right) d\mu < +\infty \right\}.$$

*The space $\mathscr{O}_\mu^\Phi$ is called Orlicz space.*

The set $\mathscr{O}_\mu^\Phi$ is a vector space. Moreover, for each $u \in \mathscr{O}_\mu^\Phi$ there exists $\beta > 0$ such that

$$\beta u \in \mathscr{B}_\Phi = \left\{ v \in \mathscr{J}_\mu^\Phi : \int_\Omega \Phi\left(|v|\right) d\mu \leqslant 1 \right\}, \tag{9}$$

where $\mathscr{B}_\Phi$ is a circled solid subset of $\mathscr{J}_\mu^\Phi$ . This property motivates the following

**Definition 2.** *We define a functional on the Orlicz space $\mathscr{O}_\mu^\Phi$ by setting*

$$\mathscr{N}_\Phi\left(u\right) = \inf\left\{ k > 0 : \tfrac{1}{k} u \in \mathscr{B}_\Phi \right\} = \inf\left\{ k > 0 : \int_\Omega \Phi\left(\left|\tfrac{u}{k}\right|\right) d\mu \leqslant 1 \right\}. \tag{10}$$

*We say that $\mathscr{N}_\Phi : \mathscr{O}_\mu^\Phi \to \mathbb{R}_+$ is the gauge norm of the Orlicz space $\mathscr{O}_\mu^\Phi$.*

It is readily seen that $\left(\mathscr{O}_\mu^\Phi, \mathscr{N}_\Phi\right)$ is a Banach space when $\mu-$a.e. equal functions are identified. Besides the gauge norm, the space $\mathscr{O}_\mu^\Phi$ can be endowed with another norm functional.

**Definition 3.** *For each $u \in \mathscr{O}_\mu^\Phi$ we set*

$$\|u\|_\Phi = \sup\left\{\int_\Omega |uv|\, d\mu : v \in \mathscr{O}_\mu^\Phi \ \ \text{such that} \ \int_\Omega \Psi\left(|v|\right) d\mu \leqslant 1\right\}, \tag{11}$$

*where $\Psi : \mathbb{R} \to \mathbb{R}_+^*$ is the complementary function to $\Phi$, defined by setting*

$$\forall\, y \in \mathbb{R} : \Psi\left(y\right) = \sup_{x \geqslant 0}\left\{x\left|y\right| - \Phi\left(x\right)\right\}.$$

*The functional $\|\cdot\|_\Phi$ is called Orlicz norm.*

One can prove that $(\mathscr{O}_\mu^\Phi, \|\cdot\|_\Phi)$ is a Banach space when $\mu$-a.e. equal functions are identified, and that the two norms $\|\cdot\|_\Phi$ and $\mathscr{N}_\Phi$ are equivalent. Moreover, there is an useful relationship between the Orlicz and gauge norms, which will be used in the following to obtain a fundamental estimate, namely

$$\forall\, u \in \mathscr{O}_\mu^\Phi : \mathscr{N}_\Phi\left(u\right) \leqslant \|u\|_\Phi \leqslant 2\mathscr{N}_\Phi\left(u\right). \tag{12}$$

We end up this section by recalling a result, due to Krasnoselskii and Rutickii, which enables to compute the Orlicz norm $\|\cdot\|_\Phi$.

**Theorem 2.** *Assume that $(\Phi, \Psi)$ be a complementary pair of Young functions such that $\Phi(x) = 0$ if and only if $x = 0$ where $\Phi$ is strictly increasing. Then*

$$\forall\, u \in \mathscr{O}_\mu^\Phi : \|u\|_\Phi = \inf_{k>0}\left\{\frac{1}{k}\left(1 + \int_\Omega \Phi\left(ku\right) d\mu\right)\right\}, \tag{13}$$

*namely the Orlicz norm $\|\cdot\|_\Phi$ is given in terms of $\Phi$ alone.*

*Proof.* See [17,24]. $\square$

This nice alternative formula will be used later on to estimate from below the Orlicz norm.

## 3. Recalls from nonsmooth critical point theory

Let us briefly recall from [8] two basic definitions in a very general framework.

**Definition 4.** *Let $(\mathscr{X}, d)$ be a metric space, $f : \mathscr{X} \to \mathbb{R}$ a continuous function and $u \in \mathscr{X}$. We denote by $|df|(u)$ the supremum of $\sigma \in [0, +\infty[$ such that there exist $\delta > 0$ and a continuous map*

$$\mathscr{H} : B_\delta(u) \times [0, \delta] \to \mathscr{X}$$

*such that for all $(v, t) \in B_\delta(u) \times [0, \delta]$*

$$d(\mathscr{H}(v, t), v) \leqslant t, \quad f(\mathscr{H}(v, t)) \leqslant f(v) - \sigma t.$$

*We say that the extended real number $|df|(u)$ is the weak slope of $f$ at $u$.*

If $\mathscr{X}$ is normed with $\|\cdot\|_{\mathscr{X}}$ and $f$ is of class $C^1$, then $|df|(u) = \|df(u)\|_{\mathscr{X}}$.

**Definition 5.** *Let $(\mathscr{X}, d)$ be a metric space, $f : \mathscr{X} \to \mathbb{R}$ a continuous function and $u \in \mathscr{X}$. We say that $u$ is a critical point of $f$ if $|df|(u) = 0$.*

Let us now return to our concrete problem choosing the space $\mathscr{X} = H_0^1(\Omega)$ and $f = f_\varphi$. It is easily verified that $f_\varphi$ is continuous.

**Definition 6.** *We say that $u$ is a weak solution of* (7) *if $u \in H_0^1(\Omega)$ and*

$$-\sum_{i,j=1}^{2} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{2} D_s a_{ij}(x,u)D_i u D_j u = p|u|^{p-2}u\, e^{|u|^p} + \varphi$$

*in the distributional space $\mathscr{D}'(\Omega)$.*

**Proposition 1.** *Let $u \in H_0^1(\Omega)$ be such that $\left|df_\varphi\right|(u) < +\infty$. Then*

$$w_u = -\sum_{i,j=1}^{2} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum_{i,j=1}^{2} D_s a_{ij}(x,u)D_i u D_j u - p|u|^{p-2}u\, e^{|u|^p} - \varphi$$

*belongs to $H^{-1}(\Omega)$ and*

$$\|w_u\|_{-1,2} \leqslant \left|df_\varphi\right|(u).$$

*In particular, each critical point of $f_\varphi$ is a weak solution to our problem.*

*Proof.* See [8, Theorem 2.1.3]. □

We now introduce a variant of the classical Palais–Smale condition that is more suitable to our nonsmooth context.

**Definition 7.** *A sequence $(u_h)$ in $H_0^1(\Omega)$ is said to be a concrete Palais–Smale sequence at level $c \in \mathbb{R}$ (*$(\mathrm{CPS})_c$-sequence*, in short) for $f_\varphi$, if $f_\varphi(u_h) \to c$,*

$$\sum_{i,j=1}^{2} D_s a_{ij}(x,u_h)D_i u_h D_j u_h \in H^{-1}(\Omega)$$

*eventually as $h \to \infty$ and*

$$-\sum_{i,j=1}^{2} D_j(a_{ij}(x,u_h)D_i u_h) + \frac{1}{2}\sum_{i,j=1}^{2} D_s a_{ij}(x,u_h)D_i u_h D_j u_h - p|u_h|^{p-2}u_h e^{|u_h|^p} \to 0,$$

*strongly in $H^{-1}(\Omega)$. We say that $f_\varphi$ satisfies the concrete Palais–Smale condition at level $c$, if every $(\mathrm{CPS})_c$-sequence for $f_\varphi$ admits a strongly convergent subsequence in $H_0^1(\Omega)$.*

It is easy to see that the validity of the $(\mathrm{CPS})_c$ condition implies the validity of the classical Palais–Smale condition ($(\mathrm{PS})_c$).

In the next theorem, we recall a generalization due to Struwe [21] of the classical perturbation argument for dealing with problems with broken symmetry, here adapted to our nonsmooth framework.

**Theorem 3.** *Let $\mathscr{X}$ be a Hilbert space endowed with a norm $\|\cdot\|_{\mathscr{X}}$ and let $f : \mathscr{X} \to \mathbb{R}$ be a continuous functional. Assume that there exists $M > 0$ such that $f$ satisfies the concrete Palais–Smale condition at each level $c \geqslant M$. Let $\mathscr{Y}$ be a finite dimensional subspace of $\mathscr{X}$ and $u^* \in \mathscr{X} \setminus \mathscr{Y}$ and set*

$$\mathscr{Y}^* = \mathscr{Y} \oplus \langle u^* \rangle, \quad \mathscr{Y}^*_+ = \{u + \lambda u^* \in \mathscr{Y}^* : u \in \mathscr{Y},\ \lambda \geqslant 0\}.$$

*Assume now that $f(0) \leqslant 0$ and that:*

(a) *there exists $R > 0$ such that:*

$$\forall u \in \mathscr{Y} : \|u\|_{\mathscr{X}} \geqslant R \Longrightarrow f(u) \leqslant f(0);$$

(b) *there exists $R^* \geqslant R$ such that:*

$$\forall u \in \mathscr{Y}^* : \|u\|_{\mathscr{X}} \geqslant R^* \Longrightarrow f(u) \leqslant f(0).$$

*Let us set*

$$\mathscr{P} = \left\{\gamma \in C(\mathscr{X}, \mathscr{X}) : \gamma \text{ odd}, \ \gamma(u) = u \text{ if } \max\{f(u), f(-u)\} \leqslant 0\right\}.$$

*Then, if*

$$c^* = \inf_{\gamma \in \mathscr{P}} \sup_{u \in \mathscr{Y}^*_+} f(\gamma(u)) > c = \inf_{\gamma \in \mathscr{P}} \sup_{u \in \mathscr{Y}} f(\gamma(u)) \geqslant M,$$

*$f$ admits at least one critical value $\overline{c} \geqslant c^*$.*

This result follows by combining [21, Ch. II, Theorem 7.1] with the nonsmooth deformation lemmas of [8]. In our concrete situation, we will use this theorem in the form of Lemma 4, which is due to P. Rabinowitz.

## 4. The perturbation argument

Let us first prove an a priori estimate for weak solutions of (7).

**Lemma 1.** *Assume that $u \in H^1_0(\Omega)$ is a weak solution of* (7). *Then*

$$\int_\Omega \left(e^{|u|^p} - 1 + c\right) dx \leqslant \sigma \left(f^2_\varphi(u) + 1\right)^{1/2},$$

*for some $\sigma > 0$ and $c > 0$.*

*Proof.* Let $k \geqslant 1$ and $\eta_k : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\eta_k(s) = \begin{cases} 0 & \text{if } s \leqslant k \\ s-k & \text{if } k \leqslant s \leqslant k+1 \\ 1 & \text{if } s \geqslant k+1. \end{cases} \tag{14}$$

For each $k \geqslant 1$, we have $f'_\varphi(u)(\eta_k(u)) = 0$. Therefore, it results

$$\begin{aligned} &\int_{\{k<u<k+1\}} \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx \\ &\qquad + \frac{1}{2} \int_\Omega \sum_{i,j=1}^{2} \eta_k(u) D_s a_{ij}(x,u) D_i u D_j u \, dx \\ &\qquad \geqslant p(k+1)^{p-1} \int_\Omega \left( e^{|u|^p} - 1 \right) dx + \int_\Omega \varphi \eta_k(u) \, dx \\ &\qquad - p(k+1)^{p-1} \left( e^{(k+1)^p} - 1 \right) \mathscr{L}^2(\Omega). \end{aligned}$$

Taking into account that $D_s a_{ij} \in L^\infty(\Omega \times \mathbb{R})$ and $|\eta_k| \leqslant 1$, inserting the expression of $f_\varphi(u)$, we find $C > 0$ and $C_{\delta,\varphi} > 0$ such that

$$\begin{aligned} &\int_{\{k<u<k+1\}} \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx \\ &\qquad + \frac{1}{2} \int_\Omega \sum_{i,j=1}^{2} \eta_k(u) D_s a_{ij}(x,u) D_i u D_j u \, dx \\ &\qquad \leqslant \frac{C}{2} \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx \\ &\qquad \leqslant C f_\varphi(u) + (1+\delta) C \int_\Omega \left( e^{|u|^p} - 1 \right) dx + C_{\delta,\varphi}, \end{aligned}$$

for each $\delta > 0$. Fixing $\delta > 0$ and choosing $k$ sufficiently large, by combining the two previous estimates we get:

$$C_k f_\varphi(u) \geqslant \int_\Omega \left( e^{|u|^p} - 1 \right) dx - C'_k,$$

for some $C_k, C'_k > 0$, which easily yields the assertion.

Let us now define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi = 1$ for $s \leqslant 1$, $\chi = 0$ for $s \geqslant 2$ and $-2 < \chi' < 0$ when $1 < s < 2$, and let us set

$$\begin{aligned} \phi(u) &= 2\sigma \left( f_\varphi^2(u) + 1 \right)^{1/2}, \\ \psi(u) &= \chi \left( \phi(u)^{-1} \int_\Omega \left( e^{|u|^p} - 1 + c \right) dx \right) \end{aligned}$$

for each $u \in H_0^1(\Omega)$. Finally, we define the modified functional by setting

$$\widetilde{f}_\varphi(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u)D_iuD_ju\,dx - \int_\Omega \left(e^{|u|^p} - 1\right)dx - \psi(u)\int_\Omega \varphi\, u\,dx.$$

The Euler's equation associated with $\widetilde{f}_\varphi$ is given by

$$-\sum_{i,j=1}^{2} D_j(a_{ij}(x,u)D_iu) + \frac{1}{2}\sum_{i,j=1}^{2} D_sa_{ij}(x,u)D_iuD_ju = \widehat{g}(x,u) \text{ in } \Omega \tag{15}$$

where we have set

$$\widehat{g}(x,u) = p|u|^{p-2}ue^{|u|^p} + \psi(u)\varphi + \psi'(u)\int_\Omega \varphi\, u\,dx.$$

Note that, by Lemma 1, if $f_\varphi'(u) = 0$, then $\widetilde{f}_\varphi(u) = f_\varphi(u)$ and $\widetilde{f}_\varphi'(u) = 0$.

*Remark 1.* If we define $\vartheta : H_0^1(\Omega) \to \mathbb{R}$ by setting:

$$\vartheta(u) = \phi(u)^{-1}\int_\Omega \left(e^{|u|^p} - 1 + c\right)dx,$$

a direct computation yields for each $v \in H_0^1 \cap L^\infty(\Omega)$:

$$\begin{aligned}
\widetilde{f}_\varphi'(u)(v) = {} & (1 + T_1(u))\int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u)D_iuD_jv\,dx \\
& + \frac{1}{2}(1 + T_1(u))\int_\Omega \sum_{i,j=1}^{2} vD_sa_{ij}(x,u)D_iuD_ju\,dx \\
& - (1 + T_2(u))\int_\Omega p|u|^{p-2}uve^{|u|^p}\,dx - (\psi(u) + T_1(u))\int_\Omega \varphi\, v\,dx,
\end{aligned}$$

where $T_1, T_2 : H_0^1(\Omega) \to \mathbb{R}$ are given by

$$\begin{aligned}
T_1(u) &= \chi'(\vartheta(u))(2\sigma)^2\vartheta(u)\phi(u)^{-2}f_\varphi(u)\int_\Omega \varphi\, u\,dx, \\
T_2(u) &= \chi'(\vartheta(u))\phi(u)^{-1}\int_\Omega \varphi\, u\,dx + T_1(u).
\end{aligned}$$

If $f_\varphi(u) \geqslant M$ and $M \to +\infty$, then $T_1(u) \to 0$ and $T_2(u) \to 0$ (see [15,16]).

The following result establishes the links between the modified functional $\widetilde{f}_\varphi$ and the original functional $f_\varphi$.

**Theorem 4.** *There exists $\widetilde{M} \in \mathbb{R}$ such that the following facts holds:*

(a) *if $u$ solves* (15) *with $\widetilde{f}_\varphi(u) \geqslant \widetilde{M}$, then $u$ solves* (7) *and $\widetilde{f}_\varphi(u) = f_\varphi(u)$;*
(b) *$\widetilde{f}_\varphi$ satisfies the concrete Palais–Smale condition at each level $c \geqslant \widetilde{M}$.*

*Proof.* By Remark 1 and Lemma 1, $(a)$ follows arguing as in [15, Theorem 2.3]. Let us now come to $(b)$. Let us first show that each $(\mathrm{CPS})_c-$sequence $(u_h) \subset H_0^1(\Omega)$ for $\widetilde{f}_\varphi$ with $c \geqslant \widetilde{M}$ is bounded in $H_0^1(\Omega)$. Let $k \geqslant 1$ and $\eta_k$ be the function defined in (14). For each $k \geqslant 1$, we have

$$\frac{\widetilde{f}'_\varphi(u_h)(\eta_k(u_h))}{\|u_h\|_{1,2}} \to 0$$

as $h \to +\infty$. In particular, it results

$$\begin{aligned}
(1+T_1(u_h)) \int_{\{k<u_h<k+1\}} \sum_{i,j=1}^{2} a_{ij}(x,u_h) D_i u_h D_j u_h \, dx& \\
+ \frac{1}{2}(1+T_1(u_h)) \int_\Omega \sum_{i,j=1}^{2} \eta_k(u_h) D_s a_{ij}(x,u_h) D_i u_h D_j u_h \, dx& \\
= (1+T_2(u_h)) \int_\Omega p|u_h|^{p-1} |\eta_k(u_h)| e^{|u_h|^p} \, dx& \\
+ (T_1(u_h)+\psi(u_h)) \int_\Omega \varphi \eta_k(u_h) \, dx + \langle w_h, \eta_k(u_h) \rangle& \\
\geqslant p(k+1)^{p-1}(1+T_2(u_h)) \int_{\{u_h \geqslant k+1\}} e^{|u_h|^p} \, dx& \\
+ (T_1(u_h)+\psi(u_h)) \int_\Omega \varphi \eta_k(u_h) \, dx + \langle w_h, \eta_k(u_h) \rangle& \\
\geqslant p(k+1)^{p-1}(1+T_2(u_h)) \int_\Omega \left( e^{|u_h|^p} - 1 \right) dx& \\
+ (T_1(u_h)+\psi(u_h)) \int_\Omega \varphi \eta_k(u_h) \, dx& \\
- 2p(k+1)^{p-1} \left( e^{(k+1)^p} - 1 \right) \mathscr{L}^2(\Omega) + \langle w_h, \eta_k(u_h) \rangle,&
\end{aligned}$$

where $w_h \to 0$ in $H^{-1}(\Omega)$. Inserting now the expression of $f_\varphi(u_h)$, we get

$$\begin{aligned}
(1+T_1(u_h)) \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u_h) D_i u_h D_j u_h \, dx& \\
+ \frac{1}{2}(1+T_1(u_h)) \int_\Omega \sum_{i,j=1}^{2} |\eta_k(u_h)| \, D_s a_{ij}(x,u_h) D_i u_h D_j u_h \, dx& \\
\geqslant \frac{p}{2}(k+1)^{p-1}(1+T_2(u_h)) \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u \, dx& \\
- p(k+1)^{p-1}(1+T_2(u_h)) f_\varphi(u_h) - p(k+1)^{p-1}(1+T_2(u_h)) \int_\Omega \varphi u_h \, dx& \\
+ (T_1(u_h)+\psi(u_h)) \int_\Omega \varphi \eta_k(u_h) \, dx& \\
- 2p(k+1)^{p-1} \left( e^{(k+1)^p} - 1 \right) \mathscr{L}^2(\Omega) + \langle w_h, \eta_k(u_h) \rangle.&
\end{aligned}$$

Taking into account that $|\eta_k| \leqslant 1$, $|\psi| \leqslant 1$ and $T_1(u_h), T_2(u_h) \to 0$ uniformly in $h$ as $\widetilde{M} \to +\infty$, by choosing $k$ large enough we find $C_k > 0$ such that

$$\begin{aligned}\nu C_k \int_\Omega |\nabla u_h|^2\, dx &\leqslant C_k \int_\Omega \sum_{i,j=1}^{2} a_{ij}(x,u) D_i u D_j u\, dx \\ &\leqslant 2p(k+1)^{p-1} f_\varphi(u_h) + 2p(k+1)^{p-1}\|\varphi\|_2\|u_h\|_2 + 2\|\varphi\|_1 \\ &\quad + 2p(k+1)^{p-1}\big(e^{(k+1)^p}-1\big)\mathscr{L}^2(\Omega) + \|w_h\|_{-1,2}\|\eta_k(u_h)\|_{1,2}.\end{aligned}$$

Since $f_\varphi(u_h) \to c$ and $w_h \to 0$ in $H^{-1}(\Omega)$, the above inequality implies that the sequence $(u_h)$ is bounded in $H_0^1(\Omega)$.

Now, let $(u_h)$ be a $(\mathrm{CPS})_c$-sequence for $\widetilde{f}_\varphi$ with $c \geqslant \widetilde{M}$. Therefore, by the previous step $(u_h)$ is bounded in $H_0^1(\Omega)$. Taking into account that the map

$$\begin{aligned} H_0^1(\Omega) &\longrightarrow H^{-1}(\Omega) \\ u &\longmapsto p\,|u|^{p-2}\, u e^{|u|^p}\end{aligned}$$

maps bounded sets of $H_0^1(\Omega)$ to relatively compact sets of $H^{-1}(\Omega)$ (see [24]), arguing as in [15, Lemma 3.3] we deduce that, up to subsequences, $(\widehat{g}(x,u_h))$ is strongly convergent in $H^{-1}(\Omega)$. Then, by [8, Theorem 2.2.4], there exists a further subsequence $(u_{h_k})$ strongly convergent in $H_0^1(\Omega)$. $\square$

## 5. The growth estimate from below

Following [16], we shall build a min–max class for $\widetilde{f}_\varphi$ and then we shall compare the growths from below and from above of the associated min–max values. Sugimura proved in [22] the following logarithmic estimate from below on the growth of the critical values $b_k$ (see Definition 8) for problem (1)

$$\forall\, k \geqslant k_0 : b_k \geqslant k\,(\log k)^{\frac{2}{p}-2}, \quad p \in (0, 1/2)\,.$$

Instead, we shall obtain the much stronger estimate:

$$\forall\, k \geqslant k_0 : b_k \geqslant k^2.$$

Let us now recall the celebrated Trudinger–Moser inequality for a smooth bounded domain $\Omega \subset \mathbb{R}^2$ in its general form: there exists $C_{TM} > 0$ such that

$$\forall\, u \in H_0^1(\Omega) : \|u\|_{1,2} \leqslant 1 \Longrightarrow \int_\Omega e^{\alpha u^2}\, dx \leqslant C_{TM}\mathscr{L}^2(\Omega),$$

for each $\alpha \in [0, 4\pi]$. See the works of Trudinger and Moser [13,25].

The following result is one of the main tools of the paper for getting the optimal estimate from below.

**Theorem 5.** *For each* $1 < p < 2$ *there exists* $0 < \vartheta \leqslant 1$ *such that*

$$\forall u \in H_0^1(\Omega) : \|u\|_{1,2} > 1 \Longrightarrow \int_\Omega \left(e^{|u|^p} - 1\right) dx \leqslant C_0 \|u\|_{1,2}^{1/\vartheta} \tag{16}$$

*where* $\vartheta$ *depends only on* $R = \|u\|_{1,2}$ *and* $C_0 > 0$ *is independent of* $p, R$.

*Proof.* Let us give an outline of the proof. First we introduce a suitable Orlicz space on the bounded domain $\Omega$, rescaling the usual Lebesgue measure in order to give an estimate from above on the gauge norm. Here the Trudinger–Moser inequality plays an important role. Then we introduce the Orlicz norm and we give an estimate from below on this norm, using (13). Finally, combining the two estimates with (12) will yield (16). Let us define a map $\Phi : \mathbb{R} \to \mathbb{R}_+$ by setting

$$\forall x \in \mathbb{R} : \Phi(x) = e^{|x|^p} - 1.$$

It is easily seen that $\Phi$ is a Young function, so that we can introduce an associated Orlicz space $\mathscr{O}_\nu^\Phi$. Let $(\Omega, \Sigma, \nu)$ be the bounded domain of $\mathbb{R}^2$ endowed with the usual $\sigma-$algebra $\Sigma$ of measurable subsets and with a suitable rescaled Lebesgue measure $\nu$, which will be determined later. Hence, by definition (10), the gauge norm $\mathscr{N}_\Phi : \mathscr{O}_\nu^\Phi \to \mathbb{R}_+$ is given by

$$\mathscr{N}_\Phi(u) = \inf\left\{k > 0 : \int_\Omega \left(e^{|\frac{u}{k}|^p} - 1\right) d\nu \leqslant 1\right\}. \tag{17}$$

We observe first that the Trudinger–Moser inequality implies

$$\int_\Omega \left(e^{|u|^p} - 1\right) dx \leqslant C'_{TM} \mathscr{L}^2(\Omega)$$

for any $u \in H_0^1(\Omega)$ such that $\|u\|_{1,2} \leqslant (4\pi)^{1/2}$ and for $C'_{TM} \geqslant C_{TM}$. Hence

$$\int_\Omega \left(e^{|\frac{u}{k}|^p} - 1\right) dx \leqslant C'_{TM} \mathscr{L}^2(\Omega) \tag{18}$$

for any $u \in H_0^1(\Omega)$ and $k > 0$ such that $\|u\|_{1,2} \leqslant k(4\pi)^{1/2}$. Inequality (18) suggests us the choice of a new measure $\nu$, defined as

$$\forall A \in \Sigma : \nu(A) = \frac{\mathscr{L}^2(A)}{C'_{TM} \mathscr{L}^2(\Omega)}.$$

Replacing $dx$ by $d\nu$, inequality (18) allows us to estimate the gauge norm from above, namely, by (17) we have

$$\forall u \in H_0^1(\Omega) : \mathscr{N}_\Phi(u) \leqslant \frac{\|u\|_{1,2}}{(4\pi)^{1/2}}. \tag{19}$$

To get the estimate from below on the gauge norm $\mathscr{N}_\Phi$, we consider now the Orlicz norm $\|\cdot\|_\Phi$ which by (13), may be written as

$$\|u\|_\Phi = \min_{k>0} \frac{1 + \int_\Omega \left(e^{k^p |u|^p} - 1\right) d\nu}{k} = \frac{1 + \int_\Omega \left(e^{k_0^p |u|^p} - 1\right) d\nu}{k_0}, \tag{20}$$

for some $k_0 > 0$ (the minimum point). Indeed, since $e^t - 1 \geqslant t$ for all $t > 0$,

$$\frac{1 + \int_\Omega \left(e^{k^p|u|^p} - 1\right) d\nu}{k} \to +\infty \quad \text{as } k \to 0^+ \text{ and as } k \to +\infty,$$

so that the infimum in (13) is actually a minimum. Therefore, by (12) and (19), to end up the proof we have to estimate from below the norm $\|\cdot\|_\Phi$. We achieve this by comparing the value of $k_0$ with $\left[\int_\Omega \left(e^{|u|^p} - 1\right) d\nu\right]^{-1}$. If

$$k_0 \leqslant \frac{1}{\int_\Omega \left(e^{|u|^p} - 1\right) d\nu},$$

we immediately get

$$\|u\|_\Phi \geqslant \int_\Omega \left(e^{|u|^p} - 1\right) d\nu. \tag{21}$$

Otherwise, if we assume

$$k_0 > \frac{1}{\int_\Omega \left(e^{|u|^p} - 1\right) d\nu},$$

we can divide the proof into 3 steps, depending on the value of

$$a = \int_\Omega \left(e^{|u|^p} - 1\right) d\nu.$$

- If $a \leqslant 1$, then there exists a $\bar{k}$, which does not depend on $u$, such that

$$\|u\|_\Phi \geqslant \frac{1}{\bar{k}} \int_\Omega \left(e^{|u|^p} - 1\right) d\nu. \tag{22}$$

Indeed, the $C^1$ map $\Theta : \mathbb{R} \to \mathbb{R}$ given by

$$\Theta(k) = \frac{1 + \int_\Omega \left(e^{k^p|u|^p} - 1\right) d\nu}{k},$$

attains its minimum in $k_0$. Then $\Theta'(k_0) = 0$, which yields

$$\begin{aligned} pk_0^p \int_\Omega |u|^p \, e^{k_0^p|u|^p} d\nu &= 1 + \int_\Omega \left(e^{k_0^p|u|^p} - 1\right) d\nu \\ &\leqslant 1 + k_0^p \int_\Omega |u|^p \, e^{k_0^p|u|^p} d\nu. \end{aligned}$$

Therefore, it is readily seen that

$$pk_0^{p-1} \int_\Omega \left(e^{|u|^p} - 1\right) d\nu \leqslant \Theta(k_0) \leqslant \frac{p}{p-1} \frac{1}{k_0},$$

since $k_0 \geqslant 1$ by $a \leqslant 1$. In particular, we obtain:

$$\frac{1}{\int_\Omega \left(e^{|u|^p} - 1\right) d\nu} \geqslant (p-1)\, k_0^p.$$

Since $k_0 > \frac{1}{\int_\Omega (e^{|u|^p}-1)d\nu}$, we get the following upper bound on $k_0$

$$k_0 \leqslant \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} = \bar{k}.$$

Inserting this inequality in (20) we obtain (22).

- If $a > 1$ and $k_0 \geqslant 1$ we can repeat the proof as in the case $a \leqslant 1$.
- If $a > 1$ and $k_0 < 1$, there are only two possibilities: either

$$\int_\Omega \left(e^{|u|^p} - 1\right)d\nu \leqslant C, \tag{23}$$

where $C > 0$ is a constant independent of $R$, or there exists $\vartheta < 1$, which depends only on $R = \|u\|_{1,2}$, in such a way that

$$k_0 < \frac{1}{\left[\int_\Omega \left(e^{|u|^p} - 1\right) d\nu\right]^\vartheta}. \tag{24}$$

We shall prove this alternative later. Relation (24) implies

$$\|u\|_\Phi > \frac{1}{k_0} > \left[\int_\Omega \left(e^{|u|^p} - 1\right)d\nu\right]^\vartheta, \tag{25}$$

while (23) yields (16) directly, for all $1 > \vartheta > 0$. Then, by (21), (22) and (25), for some $C > 0$

$$\|u\|_\Phi^{1/\vartheta} \geqslant \frac{C}{C_{TM}\mathscr{L}^2(\Omega)} \int_\Omega \left(e^{|u|^p} - 1\right)dx, \tag{26}$$

where $\vartheta = \vartheta(R) \leqslant 1$ depends only on $R = \|u\|_{1,2}$. On the other hand, combining (12) and (19) yields

$$\|u\|_\Phi \leqslant \frac{2}{(4\pi)^{1/2}} \|u\|_{1,2} \leqslant \|u\|_{1,2}. \tag{27}$$

The estimate (26) on $\|u\|_\Phi$, together with (27), imply (16). To end up the proof of the theorem, it remains to show that either (23) or (24) is verified. Observe that

$$a^{-\vartheta} = \left(\int_\Omega \left(e^{|u|^p} - 1\right)d\nu\right)^{-\vartheta} \to 1^- \quad \text{as } \vartheta \to 0^+, \tag{28}$$

depending only on $R = \|u\|_{1,2}$. Indeed, the Trudinger–Moser inequality yields, after some computations,

$$\begin{aligned} 1 < a &\leqslant \int_{\left\{|u|^{p-2}\|u\|_{1,2}^2 \leqslant 1\right\}} \left(e^{\frac{|u|^2}{\|u\|_{1,2}^2}} - 1\right)d\nu \\ &\quad + \int_{\left\{|u|^{p-2}\|u\|_{1,2}^2 > 1\right\}} \left(e^{|u|^p} - 1\right)d\nu \\ &\leqslant 1 + ce^{R^{\frac{2p}{2-p}}}, \end{aligned} \tag{29}$$

where $c > 0$ is a constant independent of $R$. Inequality (29) yields (28) directly.

Therefore, it suffices to show that for any $R > 0$ either (23) holds or there exists a constant $\varepsilon = \varepsilon(R) \in (0, 1)$ such that

$$\forall u \in H_0^1(\Omega): \ \|u\|_{1,2} = R \Longrightarrow k_0 \leqslant 1 - \varepsilon. \tag{30}$$

By (28), if (30) is verified then inequality (24) holds. Let us first show that

$$\int_\Omega \left(e^{k^p|u|^p} - 1\right) d\nu \to \int_\Omega \left(e^{|u|^p} - 1\right) d\nu \quad \text{as } k \to 1^-. \tag{31}$$

Let $k = 1 - \eta$, with $\eta \to 0^+$. Then we have

$$\begin{aligned} &\left| \int_\Omega \left(e^{|u|^p} - 1\right) d\nu - \int_\Omega \left(e^{k^p|u|^p} - 1\right) d\nu \right| \\ &\qquad \leqslant \int_\Omega e^{|u|^p} \left\{1 - e^{-\eta p|u|^p}\right\} d\nu \\ &\qquad \leqslant \left\{ \int_\Omega e^{2|u|^p} d\nu \right\}^{1/2} \cdot 2\eta p \cdot \|u\|_{2p}^p . \end{aligned} \tag{32}$$

The last integral term in inequality (32) can be estimated as in (29), obtaining (31). Analogously one can show that

$$\int_\Omega |u|^p \, e^{k^p|u|^p} d\nu \to \int_\Omega |u|^p \, e^{|u|^p} d\nu \quad \text{as } k \to 1^-. \tag{33}$$

Let us assume now that (30) is not verified. Therefore, recalling that $k_0 < 1$, there exists $R_0 > 0$ such that for any $\varepsilon \in (0, 1)$ there exists $u_\varepsilon \in H_0^1(\Omega)$ with $\|u_\varepsilon\|_{1,2} = R_0$, such that $1 > k_0 > 1 - \varepsilon$. By definition, $\Theta'(k_0) = 0$, so that

$$pk_0^p \int_\Omega |u_\varepsilon|^p \, e^{k_0^p|u_\varepsilon|^p} d\nu \leqslant 1 + \int_\Omega \left(e^{|u_\varepsilon|^p} - 1\right) d\nu = 1 + a.$$

Therefore

$$\begin{aligned} 1 + a &\geqslant p\,(1 - \varepsilon)^p \int_\Omega |u_\varepsilon|^p \, e^{k_0^p|u_\varepsilon|^p} d\nu \\ &\geqslant p\,(1 - p\varepsilon) \left\{ \int_\Omega |u_\varepsilon|^p \, e^{|u_\varepsilon|^p} d\nu - \varepsilon C\,(R_0) \right\} \\ &= (p - p^2\varepsilon)\,(a - \varepsilon C\,(R_0)) \end{aligned}$$

by (33), which implies that

$$a \leqslant \frac{1 + \left(\varepsilon p - \varepsilon^2 p^2\right) C\,(R_0)}{-p^2\varepsilon + p - 1} \tag{34}$$

for $0 < \varepsilon < \frac{p-1}{p^2}$. From (34) one can obtain the following upper bound on $a$:

$$a < \frac{4}{p - 1} \qquad \text{if} \quad 0 < \varepsilon < \min\left\{ \frac{1}{pC\,(R_0)}, \frac{p-1}{p^2}, \frac{p-1}{2p^2}, \frac{1}{p} \right\};$$

hence, if (30) is not verified, (23) holds. Let us assume now that (30) holds. By (28) there exists a $\vartheta = \vartheta(R)$ with $0 < \vartheta < 1$, such that $a^{-\vartheta} > k_0$, that is (24). $\square$

*Remark 2.* Observe that if (16) holds with $\vartheta$, then it holds for any $0 < \vartheta' < \vartheta$, since $R > 1$. Therefore, from now on we can assume that $0 < \vartheta < 1/4$ without loss of generality. The reason of this choice will be explained later.

Let now $(u_k, \lambda_k) \subset H_0^1(\Omega) \times \mathbb{R}$ be (orthonormalized) sequence of solutions to

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and define recursively

$$\mathscr{Y}_0 = \langle u_0 \rangle, \quad \forall\, k \geqslant 1 : \mathscr{Y}_{k+1} = \mathscr{Y}_k \oplus \mathbb{R}u_{k+1}.$$

Since each $\mathscr{Y}_k$ is finite dimensional, one can find $\beta_1, \beta_2, \beta_3 > 0$ such that

$$\forall\, u \in \mathscr{Y}_k : \widetilde{f}_\varphi(u) \leqslant \beta_1 \|u\|_{1,2}^2 - \beta_2 \|u\|_{1,2}^q - \beta_3,$$

for each $q > 2$. In particular, for each $k \in \mathbb{N}$ there exists $R_k > 0$ such that

$$\|u\|_{1,2} \geqslant R_k \Longrightarrow \widetilde{f}_\varphi(u) \leqslant \widetilde{f}_\varphi(0) \leqslant 0$$

for all $u \in \mathscr{Y}_k$ and $R_k \leqslant R_{k+1}$ .

**Definition 8.** *For each $k \in \mathbb{N}$ set $D_k = \mathscr{Y}_k \cap B(0, R_k)$,*

$$\Gamma_k = \left\{ \gamma \in C(D_k, H_0^1(\Omega)) : \gamma \text{ odd and } \gamma\big|_{\partial B(0,R_k)} = Id \right\},$$

*and*

$$b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \widetilde{f}_\varphi(\gamma(u)).$$

**Lemma 2 (**Intersection lemma**).** *For any $\gamma \in \Gamma_k$ and each $R < R_k$*

$$\forall\, k \geqslant 1 : \gamma\,(D_k) \cap \partial B(0, R) \cap \mathscr{Y}_{k-1}^{\perp} \neq \emptyset. \tag{35}$$

*Proof.* See [16, Lemma 1.44]. $\square$

Observe that for all $q > 2$ and each $a_1 > 0$ there exists $a_2 > 0$ with

$$e^{|s|^p} - 1 \geqslant a_1 |s|^q - a_2 \tag{36}$$

for each $s \in \mathbb{R}$.

**Lemma 3.** *There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that*

$$\forall\, k \geqslant k_0 : b_k \geqslant \beta k^2.$$

*Proof.* Let us first note that we have

$$\forall\, u \in H_0^1(\Omega) : \widetilde{f}_\varphi(u) \geqslant \mathscr{J}_\varphi(u)$$

where we have set

$$\mathscr{J}_\varphi(u) = \frac{\nu}{2}\int_\Omega |Du|^2\,dx - \int_\Omega \left(e^{|u|^p} - 1\right)dx - \psi(u)\int_\Omega \varphi\, u\, dx.$$

Therefore, it suffices to get the desired estimate for values

$$b_k = \inf_{\gamma\in\Gamma_k}\max_{u\in D_k}\mathscr{J}_\varphi(\gamma(u))$$

which, for simplicity, we avoid to rename. If $\gamma \in \Gamma_k$ and $R < R_k$, by the Intersection Lemma, we find

$$w \in \gamma(D_k) \cap \partial B_R \cap \mathscr{Y}_{k-1}^\perp$$

so that

$$\max_{u\in D_k}\mathscr{J}_\varphi(\gamma(u)) \geqslant \mathscr{J}_\varphi(\gamma(w)) \geqslant \inf_{u\in\partial B_R\cap\mathscr{Y}_{k-1}^\perp}\mathscr{J}_\varphi(u)\,. \tag{37}$$

Therefore, to obtain a lower bound for $b_k$ we have to estimate $\mathscr{J}_\varphi(u)$ from below, with $u \in \partial B_R \cap \mathscr{Y}_{k-1}^\perp$ and $R < R_k$. This estimate will be obtained applying the interpolation inequality:

$$\|u\|_r \leqslant \|u\|_s^{1-a}\,\|u\|_{1,2}^a\,, \quad 1 \leqslant s \leqslant r < \infty, \quad a = 1 - \frac{s}{r}. \tag{38}$$

From now on, suppose $u \in \partial B_R \cap \mathscr{Y}_{k-1}^\perp$ and $1 < R < R_k$. First, observe that for any $\beta > 0$ there exists a constant $c = c(\beta, p) > 0$ such that

$$\forall\, t \in [0, +\infty[\, : e^{t^p} - 1 \leqslant t^\beta e^{t^p} + c.$$

Therefore, by Hölder inequality, it results

$$\begin{aligned}\int_\Omega \left(e^{|u|^p} - 1\right)dx &\leqslant \int_\Omega |u|^\beta\, e^{|u|^p}dx + c\mathscr{L}^2(\Omega)\\ &\leqslant \|u\|_{\alpha\beta}^\beta\left(\int_\Omega e^{\frac{\alpha}{\alpha-1}|u|^p}dx\right)^{\frac{\alpha-1}{\alpha}} + c_1\end{aligned}$$

for some $c_1$, where we put

$$\alpha = \frac{1-\vartheta^2}{1-4\vartheta^2} > 1, \quad \beta = \frac{3\,(1-2\vartheta)}{1-\vartheta} > 0\,;$$

combining (16) with the previous inequality, and noting that $\frac{\alpha-1}{\alpha} < 1$, we obtain

$$\begin{aligned}\int_\Omega \left(e^{|u|^p} - 1\right)dx &\leqslant \|u\|_{\alpha\beta}^\beta\left\{\int_\Omega\left(e^{\frac{\alpha}{\alpha-1}|u|^p} - 1\right)dx\right\}^{\frac{\alpha-1}{\alpha}} + c_1\\ &\leqslant \|u\|_{\alpha\beta}^\beta\, C_{\alpha,\vartheta}\, R^{\frac{\alpha-1}{\alpha\vartheta}} + c_1,\end{aligned} \tag{39}$$

where $\vartheta = \vartheta(R)$,

$$C_{\alpha,\vartheta} = c_2 \left( \frac{\alpha}{\alpha - 1} \right)^{(\alpha-1)/\alpha p \vartheta} \tag{40}$$

and $c_2 \geqslant \max\{1, C_0\}$. Note that condition $1 < R < R_k$ can be always satisfied, by choosing $R_k$ large enough. Applying now inequality (38) with

$$r = \alpha\beta = \frac{3(1+\vartheta)}{1+2\vartheta} \geqslant 2$$

and $s = 2$, we obtain

$$\begin{gathered} \|u\|_{\alpha\beta} \leqslant \|u\|_2^{1-a} \|u\|_{1,2}^a \leqslant \lambda_k^{-\frac{1-a}{2}} \|u\|_{1,2}, \\ a = 1 - \frac{2}{\alpha\beta} = \frac{1-\vartheta}{3(1+\vartheta)}, \end{gathered} \tag{41}$$

where we have used the relation

$$\forall\, u \in \mathscr{Y}_{k-1}^{\perp} : \|u\|_2 \leqslant \frac{1}{\lambda_k^{1/2}} \|u\|_{1,2}.$$

Combining (39) with (41) yields

$$\int_\Omega \left( e^{|u|^p} - 1 \right) dx \leqslant C_{\alpha,\vartheta} \frac{1}{\lambda_k^{1/\alpha}} R^{3 \frac{1-2\vartheta^2}{1-\vartheta^2}} + c_1.$$

On the other hand, using (36) we have

$$\begin{aligned} \int_\Omega \psi(u)\, \varphi u\, dx &\leqslant \|\varphi\|_2 \|u\|_2 \leqslant c \|\varphi\|_2 \|u\|_q \\ &\leqslant c \|\varphi\|_2 \frac{a_2^{1/q} \mathscr{L}^2(\Omega)^{1/q}}{a_1^{1/q}} \left\{ \frac{\int_\Omega \left( e^{|u|^p} - 1 + a_2 \right) dx}{a_2 \mathscr{L}^2(\Omega)} \right\}^{1/q} \\ &\leqslant C_\varphi \int_\Omega \left( e^{|u|^p} - 1 \right) dx + C_{1,\varphi}, \end{aligned}$$

where we can assume $C_\varphi > 1$ and $C_{1,\varphi} > 0$ without loss of generality. Hence

$$\mathscr{J}_\varphi(u) \geqslant R^2 \left[ \frac{1}{2} - \frac{C_{\alpha,\vartheta,\varphi}}{\lambda_k^{1/\alpha}} R^{\frac{1-4\vartheta^2}{1-\vartheta^2}} \right] - C_{2,\varphi} \tag{42}$$

where $C_{\alpha,\vartheta,\varphi} = C_{\alpha,\vartheta} C_\varphi$ and $C_{2,\varphi} = c_1 C_\varphi + C_{1,\varphi} > 0$. Observe that $\frac{1-4\vartheta^2}{1-\vartheta^2} > 0$ for all $0 < \vartheta < 1/2$; hence, we can choose $R = R(k)$ such that

$$\lambda_k^{1/\alpha} = \lambda_k^{\frac{1-4\vartheta^2}{1-\vartheta^2}} = 4 C_{\alpha,\vartheta,\varphi} R^{\frac{1-4\vartheta^2}{1-\vartheta^2}}.$$

Since $\lambda_k \geqslant c_4 k$ for large $k$ (being $n = 2$), $R$ is subjected to the lower bound

$$R^2 \geqslant \left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\vartheta,\varphi}} \right]^{2\alpha} k^2, \tag{43}$$

where we may assume that $0 < c_4 < 1$ without loss of generality; we remark that $\vartheta = \vartheta(k)$. Combining (42) with (43) yields the following estimate from below:

$$\mathscr{J}_\varphi(u) \geqslant \left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\vartheta,\varphi}} \right]^{2\alpha} k^2, \tag{44}$$

which holds for $k$ large enough. It remains to prove that the constant cut in the right-hand side of inequality (44), which depends on $\vartheta$, may be bounded from below uniformly. By (40), recalling that $0 < c_4 < 1$ and $C_{\alpha,\vartheta,\varphi} \geqslant 1$

$$\left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\vartheta,\varphi}} \right]^{2\alpha} \geqslant \left[ \frac{c_4}{4c_2 C_\varphi} \right]^{2\alpha} \cdot \left( \frac{\alpha - 1}{\alpha} \right)^{\dfrac{2(\alpha-1)}{p\vartheta}}$$

But $\frac{\alpha-1}{\alpha} = \frac{3\vartheta^2}{1-\vartheta^2}$ so that

$$\left( \frac{\alpha - 1}{\alpha} \right)^{\frac{2(\alpha-1)}{p\vartheta}} \to 1, \quad \left[ \frac{c_4}{4c_2 C_\varphi} \right]^{2\alpha} \to C_1 > 0$$

as $\vartheta \to 0$. Therefore, we obtain that

$$\left[ \frac{c_4^{1/\alpha}}{4C_{\alpha,\vartheta,\varphi}} \right]^{2\alpha} \geqslant C \tag{45}$$

for all $\vartheta$ small enough, where $C > 0$ is a constant independent on $\vartheta$. By (37),

$$b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \mathscr{J}_\varphi(\gamma(u)) \geqslant \inf_{u \in \partial B_R \cap \mathscr{Y}_{k-1}^\perp} \mathscr{J}_\varphi(u).$$

By combining (44) with (45), for $k$ large enough there exists $R = R(k) \in (0, R_k)$ such that for all $u \in \partial B_R \cap \mathscr{Y}_{k-1}^\perp$

$$\mathscr{J}_\varphi(u) \geqslant Ck^2,$$

and the proof is now complete. $\square$

## 6. The growth estimate from above

**Definition 9.** *We denote by $U_k$ the set of $\xi = tu_{k+1} + w$ such that*

$$0 \leqslant t \leqslant R_{k+1}, \quad w \in B(0, R_{k+1}) \cap \mathscr{Y}_k, \quad \|\xi\|_{1,2} \leqslant R_{k+1}.$$

*We denote by $\Lambda_k$ the set of $\lambda \in C(U_k, H_0^1(\Omega))$ such that*

$$\lambda_{|D_k} \in \Gamma_{k+1}, \quad \lambda|_{\partial B(0,R_{k+1}) \cup ((B(0,R_{k+1}) \setminus B(0,R_k)) \cap \mathscr{Y}_k)} = Id$$

*and we set*

$$c_k = \inf_{\lambda \in \Lambda_k} \max_{u \in U_k} \widetilde{f}_\varphi(\lambda(u)).$$

The next is our main existence tool.

**Lemma 4.** *Assume that $c_k > b_k \geqslant \widetilde{M}$ for $k$ large. If $\delta \in ]0, c_k - b_k[$ and*

$$\Lambda_k(\delta) = \left\{ \lambda \in \Lambda_k : \ \widetilde{f}_\varphi(\lambda(u)) \leqslant b_k + \delta \ \text{ for } \ u \in D_k \right\},$$

*set*

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \max_{u \in U_k} \widetilde{f}_\varphi(\lambda(u)).$$

*Then $c_k(\delta)$ is a critical value for $\widetilde{f}_\varphi$.*

*Proof.* See [15, Lemma 5.5]. Of course, in this nonsmooth framework, we apply [8, Theorem 1.1.13] instead of the deformation Lemma for smooth functionals (see e.g. Lemma 1.60 of [16]). □

**Lemma 5.** *Let $c_k = b_k$ for $k$ large. Then there exist $\gamma > 0$ and $k_1 \in \mathbb{N}$ such that*

$$\forall\, k \geqslant k_1 : b_k \leqslant \gamma k^{q/q-1}$$

*for each $q > 2$.*

*Proof.* Let $q > 2$. Following [15, Lemma 2.2], there exists $\alpha_{\varphi,q} > 0$ such that

$$\left| \widetilde{f}_\varphi(u) - \widetilde{f}_\varphi(-u) \right| \leqslant \alpha_{\varphi,q} \left\{ |\widetilde{f}_\varphi(u)|^{1/q} + 1 \right\}$$

for each $u \in H_0^1(\Omega)$. At this point argue as in [15, Lemma 5.6]. □

## 7. Proof the main result

Let us consider values of $k$ such that $c_k \geqslant b_k \geqslant \widetilde{M}$. By assertion $(a)$ of Theorem 4 the functional $\widetilde{f}_\varphi$ satisfies the concrete Palais–Smale condition at level $c_k$. Since $q/(q-1) < 2$, by combining Lemma 3 and Lemma 5 we deduce that $c_k > b_k$, so that we may apply Lemma 4 and obtain that $c_k(\delta)$ is a critical value for $\widetilde{f}_\varphi$. Therefore, by $(b)$ of Theorem 4, $f_\varphi$ admits a diverging sequence of critical values (hence of weak solutions of (7)). To cover the case of a general nonlinearity $\varphi$, it suffices to apply slight adaptations to several of the Lemmas (see [16]). □

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