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ON THE LOCATION OF SPIKES FOR THE SCHRÖDINGER EQUATION WITH ELECTROMAGNETIC FIELD

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We consider the standing wave solutions of the three dimensional semilinear Schrödinger equation with competing potential functions V and K and under the action of an external electromagnetic field B. We establish some necessary conditions for a sequence of such solutions to concentrate, in two different senses, around a given point. In the particular but important case of nonlinearities of power type, the spikes locate at the critical points of a smooth ground energy map independent of B.

Keywords: Nonlinear Schrödinger equation; least energy solutions; electromagnetic field; semi-classical limit; Clarke's subdifferential.

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1. Introduction

In this work we deal with the standing wave solutions

$$\varphi(x,t) = e^{-\frac{iV_0}{\hbar}t}u(x), \quad x \in \mathbb{R}^3, \ \ t \in \mathbb{R}^+$$

of the time-dependent Schrödinger equation with electromagnetic field

$$i\hbar\frac{\partial\varphi}{\partial t} = \left(\frac{\hbar}{i}\nabla - A(x)\right)^2 \varphi + W(x)\varphi - |\varphi|^{p-1}\varphi,$$

where the Schrödinger operator is defined as

$$\left(\frac{\hbar}{i}\nabla - A\right)^2 := -\hbar^2\Delta - \frac{2\hbar}{i}\left\langle A \mid \nabla\right\rangle + |A|^2 - \frac{\hbar}{i}\mathrm{div}\, A.$$

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Here $\hbar > 0$ is the Planck constant, $p \in (1,5)$, and the functions $W\colon \mathbb{R}^3 \to \mathbb{R}$ and $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ are, respectively, a scalar potential of the electric field $E = -\nabla W$ and a vector potential for the external electromagnetic field $B = \operatorname{curl} A$. Now, the function $u\colon \mathbb{R}^3 \to \mathbb{C}$ which appears in $\varphi(x,t)$ satisfies, more generally, a time-independent equation of the form

$$\left(\frac{\hbar}{i}\nabla - A(x)\right)^2 u + V(x)u = K(x)f(|u|^2)u, \tag{1.1}$$

where $V(x) = W(x) + V_0$, $K\colon \mathbb{R}^3 \to \mathbb{R}$ is an additional potential function, and $f\colon \mathbb{R}^+ \to \mathbb{R}$ is a suitable nonlinearity. Quite recently, under reasonable assumptions on A, V and K, the study of the existence of ground (bound) state solutions $u_\hbar$ to (1.1) and the related investigation of the semi-classical limit (the transition from Quantum to Classical Mechanics as $\hbar \to 0$), has been tackled in various contributions (see e.g. [2, 4, 6, 7, 14, 19] for the case $A \neq 0$ and [3, 11–13, 15, 23, 26] for the case $A = 0$). More precisely, it turns out that, if $z_0 \in \mathbb{R}^3$ is a non-degenerate critical point of the so called ground-energy function $\Sigma_r\colon \mathbb{R}^3 \to \mathbb{R}$ (see Definition 2.2), then for every $\hbar$ sufficiently small (1.1) admits a least energy solution $u_\hbar$ concentrating near z_0. In the opposite direction, we are interested in discussing some necessary conditions for the concentration of a sequence of bound-state solutions to (1.1) in the neighborhood of a given point z_0. In absence of the electromagnetic field, this problem has been studied in various papers (see e.g. [1, 29, 30]), mainly in the case where $f(u)$ is a power of exponent p (see also [17, 24]). It turns out that, at least in this particular situation, for the concentration to occur, z_0 has to be a critical point for the C^1 ground-energy map (see [30, Lemma 2.5])

$$\Sigma_r(z) = \frac{V^{\frac{5-p}{2p-2}}(z)}{K^{\frac{2}{p-1}}(z)}, \quad \text{for every } z \in \mathbb{R}^3. \tag{1.2}$$

On the other hand, to our knowledge, for a more general nonlinearity $f(u)$, the function $\Sigma_r(z)$ is locally Lipschitz continuous, and its further smoothness properties seem to depend on the uniqueness results for the limiting equation

$$-\Delta u + V(z)u = K(z)f(|u|^2)u, \tag{1.3}$$

where $z \in \mathbb{R}^3$ acts as a parameter. To overcome this problem, recently, the authors have provided in [28] new necessary conditions involving generalized derivatives of Σ_r such as the Clarke subdifferential or even weaker conditions, not requiring any regularity of Σ_r (see Definition 2.4).

Our purpose in this paper is to understand what happens under the presence of an external electromagnetic vector potential A, and to see whether A may influence or not the location of spikes for the solutions of (1.1). Actually, in general, this fact seems to depend on the notion of concentration that one adopts. We consider at least two ways of saying that a sequence $(u_\hbar)$ of bound-state solutions to (1.1) is peaking around a given point z_0. The first one, the most intuitive, is a *pointwise concentration* and it is precisely the one used in two papers by Wang and Zheng [29, 30]. The second is a sort of *energetic concentration* in terms of the functional associated with (1.1),

$$J_\hbar(u) = \frac{1}{2}\int_{\mathbb{R}^3} |D^\hbar u|^2 + V(x)|u|^2 dx - \int_{\mathbb{R}^3} K(x)F(|u|^2)dx,$$

where $D^\hbar = \frac{\hbar}{i}\nabla - A(x)$. Precisely, we require that

$$\lim_{\hbar\to 0} \hbar^{-3} J_\hbar(u_\hbar) = \Sigma_r(z_0).$$

As we prove in the main result, Theorem 3.1, the vector potential A might affect the location of pointwise concentration points, whereas it does not influence the energetic concentration points. In the particular but fairly significant case where f is a power nonlinearity, the above notions of concentration coincide (see Proposition 2.1), and it turns out that the peaks locate at the classical critical points of the smooth function (1.2) independent of A, thus rigorously confirming what was conjectured in [7]. In some sense, from a heuristic point of view, A tends to lurk into the complex phase factor of the solutions. We point out that, in the course of the proof of Theorem 3.1, we will derive an ad-hoc Pucci–Serrin type identity for the complex-valued solutions to (1.1) (cf. formula (3.6)). Just for the sake of simplicity, we restrict the attention to the physically relevant case of space-dimension $n = 3$.

Notations.

(1) $\Re w$ (respectively, $\Im w$) stands for the real (respectively, the imaginary) part of $w \in \mathbb{C}$.
(2) i is the imaginary unit, namely $i^2 = -1$. For $w \in \mathbb{C}$, we set $\bar{w} = \Re w - i\Im w$.
(3) The gradient of a C^1 function $f\colon \mathbb{R}^3 \to \mathbb{R}$ will be denoted by ∇f. The jacobian matrix of a C^1 function $g\colon \mathbb{R}^3 \to \mathbb{R}^3$ will be indicated by g'. The directional derivatives of f and g along a vector w will be indicated by $\frac{\partial f}{\partial w}$ and $\frac{\partial g}{\partial w}$.
(4) $\langle x \mid y\rangle$ denotes the standard scalar product in $\mathbb{R}^3$ of x and y.

2. Problem Setting and Auxiliary Results

In this section, we collect a few preliminary definitions and results that we need in order to state and prove our main achievement, Theorem 3.1. For the sake of simplicity, we rename the constant $\hbar$ into $\varepsilon > 0$. We assume that the functions

$$A\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad V\colon \mathbb{R}^3 \to \mathbb{R}, \quad K\colon \mathbb{R}^3 \to \mathbb{R}$$

are all of class C^1, K is positive and there exist $V_0 > 0$ and $K_0 > 0$ with

$$\inf_{x\in\mathbb{R}^3} V(x) = V_0 \quad \text{and} \quad \sup_{x\in\mathbb{R}^3} K(x) = K_0. \tag{2.1}$$

Moreover, the function $f\colon \mathbb{R}^+ \to \mathbb{R}$ is of class C^1, increasing, $f(0) = 0$ and

$$\lim_{s\to\infty} \frac{f(s)}{s^{\frac{p-1}{2}}} = 0 \quad \text{and} \quad 0 < \vartheta F(s) \le f(s)s \quad \text{for some } p \in (1,5) \text{ and } \vartheta > 2,$$

where $F(s) = \frac{1}{2}\int_0^s f(t)\,dt$ for $s \in \mathbb{R}^+$. In order to formulate the problem in a suitable variational setting, for every $\varepsilon > 0$, we introduce the (real) Hilbert space $\mathcal{H}^\varepsilon_{A,V}$ defined as the closure of $C_c^\infty(\mathbb{R}^3,\mathbb{C})$ with respect to scalar product

$$(u,v)_{\mathcal{H}^\varepsilon_{A,V}} := \Re\int_{\mathbb{R}^3} D^\varepsilon u\overline{D^\varepsilon v} + V(x)u\bar{v}\,dx, \quad D^\varepsilon u = \frac{\varepsilon}{i}\nabla - A(x).$$

As remarked in [14], $\mathcal{H}^\varepsilon_{A,V}$ has in general no relationships with $H^1(\mathbb{R}^3,\mathbb{C})$. However, the following *diamagnetic inequality* is well known (see e.g. [21])

$$\varepsilon|\nabla|u|(x)| \le |D^\varepsilon u(x)|, \quad \text{for every } u \in \mathcal{H}^\varepsilon_{A,V} \text{ and a.e. } x \in \mathbb{R}^3, \tag{2.2}$$

so that $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$ for any $u \in \mathcal{H}^\varepsilon_{A,V}$. Finally we recall that the Schrödinger operator is *gauge invariant*: if we replace A by $\tilde{A} = A + \nabla\chi$ for any $\chi \in C^2(\mathbb{R}^3,\mathbb{R})$, and we let $\tilde{u} = e^{\frac{i}{\varepsilon}\chi}u$, then $\operatorname{curl}\tilde{A} = \operatorname{curl} A$ and

$$\left(\frac{\varepsilon}{i}\nabla - \tilde{A}\right)\tilde{u} = e^{\frac{i}{\varepsilon}\chi}\left(\frac{\varepsilon}{i}\nabla - A\right)u,$$

so that $\|\tilde{u}\|_{\mathcal{H}^\varepsilon_{\tilde{A},V}} = \|u\|_{\mathcal{H}^\varepsilon_{A,V}}$.

Under the above assumptions, we give the following

Definition 2.1. We say that (u_ε) is a sequence of bound-state solutions to

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = K(x)f(|u|^2)u \tag{S_ε}$$

if u_ε belongs to $\mathcal{H}^\varepsilon_{A,V}$ for every $\varepsilon > 0$,

$$\sup_{\varepsilon>0}\varepsilon^{-3}\|u_\varepsilon\|^2_{\mathcal{H}^\varepsilon_{A,V}} < \infty \tag{2.3}$$

and u_ε satisfies (S_ε) on $\mathbb{R}^3$ in weak sense.

2.1. *The ground-energy functions*

Fixed $z \in \mathbb{R}^3$, we consider the functional

$$I_z(u) = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2 + V(z)|u|^2\,dx - \int_{\mathbb{R}^3}K(z)F(|u|^2)\,dx$$

associated with the limiting equation (1.3). It is readily seen that I_z is C^1 over both the spaces $H^1(\mathbb{R}^3,\mathbb{R})$ and $H^1(\mathbb{R}^3,\mathbb{C})$.

Definition 2.2. We define the real and the complex ground-state functions

$$\Sigma_r\colon \mathbb{R}^3 \to \mathbb{R} \quad \text{and} \quad \Sigma_c\colon \mathbb{R}^3 \to \mathbb{R}$$

by setting, for every $z \in \mathbb{R}^3$,

$$\Sigma_r(z) = \min_{v\in\mathcal{N}_z} I_z(v) \quad \text{and} \quad \Sigma_c(z) = \min_{v\in\tilde{\mathcal{N}}_z} I_z(v),$$

where $\mathcal{N}_z$ (respectively, $\tilde{\mathcal{N}_z}$) are the real (respectively, the complex) Nehari manifolds,

$$\mathcal{N}_z = \left\{u \in H^1(\mathbb{R}^3, \mathbb{R})\backslash\{0\} : I_z'(u)[u] = 0\right\},$$
$$\tilde{\mathcal{N}_z} = \left\{u \in H^1(\mathbb{R}^3, \mathbb{C})\backslash\{0\} : I_z'(u)[u] = 0\right\}.$$

Here $I_z'(u)[v]$ stands for the directional derivative of I_z at u along v.

We denote by $S_r(z)$ the set of positive radial solutions up to translations to (1.3) at the energy level $\Sigma_r(z)$. As the next lemma claims, the map Σ_r enjoys some useful regularity properties (see [30]).

Lemma 2.1. *The following facts hold*:

(i) Σ_r *is locally Lipschitz continuous*;

(ii) *the directional derivatives from the left and the right of* Σ_r *at every point* $z \in \mathbb{R}^3$ *along any* $w \in \mathbb{R}^3$ *exist and it holds*

$$\left(\frac{\partial \Sigma_r}{\partial w}\right)^-(z) = \sup_{v \in S_r(z)} \langle \nabla_z I_z(v) \mid w \rangle,$$
$$\left(\frac{\partial \Sigma_r}{\partial w}\right)^+(z) = \inf_{v \in S_r(z)} \langle \nabla_z I_z(v) \mid w \rangle.$$

Explicitly, we have

$$\left(\frac{\partial \Sigma_r}{\partial w}\right)^-(z) = \sup_{v \in S_r(z)} \left[\frac{\partial V}{\partial w}(z) \int_{\mathbb{R}^3} \frac{|v|^2}{2}\, dx - \frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^3} F(|v|^2)\, dx\right],$$
$$\left(\frac{\partial \Sigma_r}{\partial w}\right)^+(z) = \inf_{v \in S_r(z)} \left[\frac{\partial V}{\partial w}(z) \int_{\mathbb{R}^3} \frac{|v|^2}{2}\, dx - \frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^3} F(|v|^2)\, dx\right],$$

for every $z, w \in \mathbb{R}^3$.

The next result will turn out to be pretty useful along the proof of our main theorem. We stress that it contains, as a particular case, [19, Lemma 7].

Lemma 2.2. *The following facts hold*:

(i) $\Sigma_c(z) = \Sigma_r(z)$, *for every* $z \in \mathbb{R}^3$;

(ii) *if* $U_z \colon \mathbb{R}^3 \to \mathbb{C}$ *is a least energy solution of problem* (1.3), *then*

$$|\nabla |U_z|(x)| = |\nabla U_z(x)| \quad \textit{and} \quad \Re\big(i\bar{U}_z(x)\nabla U_z(x)\big) = 0,$$

for a.e. $x \in \mathbb{R}^3$;

(iii) *there exist* $\omega \in \mathbb{R}$ *and a real-valued least energy solution* u_z *of problem* (1.3) *with*

$$U_z(x) = e^{i\omega} u_z(x), \quad \textit{for a.e. } x \in \mathbb{R}^3. \tag{2.4}$$

Proof. Fix $z \in \mathbb{R}^3$. For the sake of convenience, we introduce the functionals

$$T(u) = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx,$$
$$P_z(u) = \int_{\mathbb{R}^3} \left[K(z)F(|u|^2) - \frac{1}{2}V(z)|u|^2 \right] dx.$$

Observe that $I_z(u) = \frac{1}{2}T(u) - P_z(u)$. Consider the following minimization problems

$$\sigma_r(z) = \min\{T(u) \colon u \in H^1(\mathbb{R}^3, \mathbb{R}),\ P_z(u) = 1\},$$
$$\sigma_c(z) = \min\{T(u) \colon u \in H^1(\mathbb{R}^3, \mathbb{C}),\ P_z(u) = 1\}.$$

Note that, obviously, there holds $\sigma_c(z) \leq \sigma_r(z)$. If we denote by $u_\star$ the Schwarz symmetric rearrangement (see e.g. [3, 21]) of the positive real valued function $|u| \in H^1(\mathbb{R}^3, \mathbb{R})$, then, Cavalieri's principle yields

$$\int_{\mathbb{R}^3} F(|u_\star|^2) \, dx = \int_{\mathbb{R}^3} F(|u|^2) \, dx \quad \text{and} \quad \int_{\mathbb{R}^3} |u_\star|^2 \, dx = \int_{\mathbb{R}^3} |u|^2 \, dx,$$

which entails $P_z(u_\star) = P_z(u)$. Moreover, by the Polya–Szegö inequality, we have

$$T(u_\star) = \int_{\mathbb{R}^3} |\nabla u_\star|^2 \, dx \leq \int_{\mathbb{R}^3} |\nabla |u||^2 \, dx \leq \int_{\mathbb{R}^3} |\nabla u|^2 \, dx = T(u),$$

where the second inequality follows by (2.2) with $A = 0$ and $\varepsilon = 1$. Therefore, one can compute $\sigma_c(z)$ by minimizing over the subclass of positive, radially symmetric and radially decreasing functions $u \in H^1(\mathbb{R}^3, \mathbb{R})$. As a consequence, $\sigma_r(z) \leq \sigma_c(z)$. In conclusion, $\sigma_r(z) = \sigma_c(z)$. Observe now that

$$\Sigma_r(z) = \min \{I_z(u) \colon u \in H^1(\mathbb{R}^3, \mathbb{R}) \setminus \{0\} \text{ is a solution to } (1.3)\},$$
$$\Sigma_c(z) = \min \{I_z(u) \colon u \in H^1(\mathbb{R}^3, \mathbb{C}) \setminus \{0\} \text{ is a solution to } (1.3)\}.$$

The above equations hold since any nontrivial real (respectively, complex) solution of (1.3) belongs to $\mathcal{N}_z$ (respectively, $\tilde{\mathcal{N}_z}$) and, conversely, any solution of $\Sigma_r(z)$ (respectively, $\Sigma_c(z)$) produces a nontrivial solution of (1.3). Moreover, it follows from an easy adaptation of [3, Theorem 3, p. 331] that $\Sigma_r(z) = \sigma_r(z)$ as well as $\Sigma_c(z) = \sigma_c(z)$. In conclusion,

$$\Sigma_r(z) = \sigma_r(z) = \sigma_c(z) = \Sigma_c(z),$$

which proves (i). To prove (ii), let $U_z \colon \mathbb{R}^3 \to \mathbb{C}$ be a least energy solution to problem (1.3). There holds $|\nabla |U_z|| \leq |\nabla U_z|$. Assume by contradiction that

$$\mathcal{L}^3(\{x \in \mathbb{R}^3 \colon |\nabla |U_z|(x)| < |\nabla U_z(x)|\}) > 0,$$

where $\mathcal{L}^3$ is the Lebesgue measure in $\mathbb{R}^3$. Then we get $P_z(|U_z|) = P_z(U_z)$ and

$$\sigma_r(z) \leq \int_{\mathbb{R}^3} |\nabla |U_z||^2 \, dx < \int_{\mathbb{R}^3} |\nabla U_z|^2 \, dx = \sigma_c(z),$$

which is a contradiction. The second assertion in (ii) follows by a direct computation. Indeed, a.e. in $\mathbb{R}^3$, we have

$$|\nabla|U_z|| = |\nabla U_z| \quad \text{if and only if } \Re U_z \nabla(\Im U_z) = \Im U_z \nabla(\Re U_z).$$

If this last condition holds, in turn, a.e. in $\mathbb{R}^3$ we have

$$\bar{U}_z \nabla U_z = \Re U_z \nabla(\Re U_z) + \Im U_z \nabla(\Im U_z),$$

which implies the desired assertion. Finally, the representation formula of (iii) is an immediate consequence of (ii), since one obtains $U_z = e^{i\omega}|U_z|$ for some $\omega \in \mathbb{R}$. □

2.2. *Generalized gradients*

Assume that $f\colon \mathbb{R}^3 \to \mathbb{R}$ is a locally Lipschitz continuous function. For the reader's convenience, we recall that the Clarke subdifferential (or generalized gradient) of f at a point z (cf. [8]) is defined as

$$\partial_C f(z) = \left\{\eta \in \mathbb{R}^3\colon f^0(z, w) \geq \langle \eta \mid w \rangle\,, \text{ for every } w \in \mathbb{R}^3\right\},$$

where $f^0(z, w)$ is the Clarke derivative of f at z along the direction w, defined as

$$f^0(z; w) = \limsup_{\substack{\xi \to z \\ \lambda \to 0^+}} \frac{f(\xi + \lambda w) - f(\xi)}{\lambda}.$$

From [8, Proposition 2.3.1] we learn that $\partial_C f(z)$ is nonempty, convex and

$$\partial_C(-f)(z) = -\partial_C f(z), \quad \text{for every } z \in \mathbb{R}^3. \tag{2.5}$$

In light of (i) in Lemma 2.1, we are allowed to give the following

Definition 2.3. We denote by $\mathfrak{S} \subset \mathbb{R}^3$ the set of critical points of the function Σ_r in the sense of the Clarke subdifferential, namely

$$\mathfrak{S} := \left\{z \in \mathbb{R}^3\colon 0 \in \partial_C \Sigma_r(z)\right\}.$$

Now, for $z \in \mathbb{R}^3$, we consider the gauge invariant functional $J_z\colon H^1(\mathbb{R}^3, \mathbb{C}) \to \mathbb{R}$

$$J_z(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left|\left(\frac{1}{i}\nabla - A(z)\right) u\right|^2 + V(z)|u|^2\, dx - \int_{\mathbb{R}^3} K(z) F(|u|^2)\, dx,$$

associated with the limiting equation

$$\left(\frac{1}{i}\nabla - A(z)\right)^2 u + V(z)u = K(z) f(|u|^2)u.$$

We denote by $G_c(z)$ the set of the nontrivial solutions $v\colon \mathbb{R}^3 \to \mathbb{C}$, up to translations, of the above limiting problem with bounded, but not necessarily least, energy. Moreover, we introduce the linear map $\Upsilon_z\colon \mathbb{R}^3 \to \mathbb{R}$, defined as

$$\Upsilon_z(x) := \sum_{j=1}^{3} A_j(z) x_j, \quad \text{for every } x \in \mathbb{R}^3.$$

Apparently, for every $z \in \mathbb{R}^3$, there holds $\nabla \Upsilon_z(x) = A(z)$. It is readily seen that for every $v \in G_c(z)$ we can write $v = e^{i\Upsilon_z} U_z$, where U_z is a (possibly complex-valued) solution to problem (1.3).

Definition 2.4. Let $z \in \mathbb{R}^3$. For every $w \in \mathbb{R}^3$ we define $\Gamma_z^-(w)$ and $\Gamma_z^+(w)$ by

$$\Gamma_z^-(w) := \sup_{v \in G_c(z)} \langle \nabla_z J_z(v) \mid w \rangle \quad \text{and} \quad \Gamma_z^+(w) := - \inf_{v \in G_c(z)} \langle \nabla_z J_z(v) \mid w \rangle,$$

where ∇_z is the gradient with respect to z. Explicitly, for every $w \in \mathbb{R}^3$,

$$\begin{aligned}
\Gamma_z^-(w) = &\sup_{\substack{v = e^{i\Upsilon_z} U_z \\ v \in G_c(z)}} \left[\left\langle \frac{\partial A}{\partial w}(z) \middle| \int_{\mathbb{R}^3} \Re(i\bar{U}_z \nabla U_z)\, dx \right\rangle \right. \\
&\left. + \frac{\partial V}{\partial w}(z) \int_{\mathbb{R}^3} \frac{|U_z|^2}{2}\, dx - \frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^3} F(|U_z|^2)\, dx \right], \\
\Gamma_z^+(w) = - &\inf_{\substack{v = e^{i\Upsilon_z} U_z \\ v \in G_c(z)}} \left[\left\langle \frac{\partial A}{\partial w}(z) \middle| \int_{\mathbb{R}^3} \Re(i\bar{U}_z \nabla U_z)\, dx \right\rangle \right. \\
&\left. + \frac{\partial V}{\partial w}(z) \int_{\mathbb{R}^3} \frac{|U_z|^2}{2}\, dx - \frac{\partial K}{\partial w}(z) \int_{\mathbb{R}^3} F(|U_z|^2)\, dx \right].
\end{aligned}$$

Notice that

$$\partial \Gamma_z^\pm(0) = \left\{ \eta \in \mathbb{R}^3 \colon \Gamma_z^\pm(w) \geq \langle \eta \mid w \rangle, \ \text{ for every } w \in \mathbb{R}^3 \right\},$$

where $\partial \Gamma_z^\pm(0)$ is the subdifferential of the convex function $\Gamma_z^\pm$ at zero. We set

$$\mathfrak{S}^* := \left\{ z \in \mathbb{R}^3 \colon 0 \in \partial \Gamma_z^-(0) \cap \partial \Gamma_z^+(0) \right\}$$

and we say that $\mathfrak{S}^*$ is the set of weak-concentration points for problem (S_ε).

2.3. *Concentration of bound-state solutions*

We now introduce two (gauge invariant) notions of concentration for a sequence of bound-states solutions of (S_ε) around a given point.

Definition 2.5. Let $z_0 \in \mathbb{R}^3$ and assume that $(u_{\varepsilon_h}) \subset \mathcal{H}_{A,V}^{\varepsilon_h}$ is a sequence of bound-state solutions to problem (S_ε). We say that

(i) z_0 is a *concentration point* for (u_{ε_h}) if $|u_{\varepsilon_h}(z_0)| \geq \varrho > 0$ and for every $\eta > 0$ there exist $\rho > 0$ and $h_0 \geq 1$ such that

$$|u_{\varepsilon_h}(x)| \leq \eta, \quad \text{for every } h \geq h_0 \text{ and } |x - z_0| \geq \varepsilon_h \rho.$$

The set of such points will be denoted by $\mathcal{C} \subset \mathbb{R}^3$;

(ii) z_0 is an *energy-concentration point* if, in addition

$$\lim_{h \to \infty} \varepsilon_h^{-3} J_{\varepsilon_h}(u_{\varepsilon_h}) = \Sigma_r(z_0). \tag{$*$}$$

The set of such points will be denoted by $\mathcal{E} \subset \mathbb{R}^3$.

For instance, if $K \equiv 1$, f is a power, z_0 is a minimum point of V and (u_{ε_h}) is a sequence of least-energy solutions to (S_ε), then $z_0 \in \mathcal{E} \neq \emptyset$ (cf. [19, Lemma 3]).

Next we see that in the case of power nonlinearities

$$f(u) = \lambda u^{\frac{p-1}{2}} \quad \text{for some } p \in (1,5) \text{ and } \lambda > 0, \tag{2.6}$$

the above notions (i) and (ii) coincide.

Proposition 2.1. *Let f be as in* (2.6). *Then $\mathcal{E} = \mathcal{C}$.*

Proof. We can prove the proposition under the mere assumption (*). Consider $v_h(x) = u_{\varepsilon_h}(z_0 + \varepsilon_h x)$. Then $(|v_h|)$ converges to some $\tilde{v} \geq 0$ weakly in $H^1(\mathbb{R}^3, \mathbb{R})$ and strongly in $L^q_{\text{loc}}(\mathbb{R}^3, \mathbb{R})$ for $2 \leq q < 6$ (see Step I in the proof of Theorem 3.1). By Kato's inequality [27, Theorem X.33], we get

$$\int_{\mathbb{R}^3} K(z_0 + \varepsilon_h x)|v_h|^p \tilde{v}\, dx \geq \int_{\mathbb{R}^3} \nabla |v_h| \nabla \tilde{v} + V(z_0 + \varepsilon_h x)|v_h| \tilde{v}\, dx$$

which, as $h \to \infty$, yields,

$$\int_{\mathbb{R}^3} K(z_0)|\tilde{v}|^{p+1}\, dx \geq \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 + V(z_0)|\tilde{v}|^2\, dx.$$

Therefore, there exists $\vartheta \in (0,1]$ such that $\vartheta \tilde{v} \in \mathcal{N}_{z_0}$. As a consequence,

$$\begin{aligned}
\Sigma_r(z_0) &\leq \vartheta^2 \left(\frac{1}{2} - \frac{1}{p+1}\right) \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 + V(z_0)|\tilde{v}|^2\, dx \\
&\leq \left(\frac{1}{2} - \frac{1}{p+1}\right) \liminf_{h\to\infty} \int_{\mathbb{R}^3} |\nabla |v_h||^2 + V(z_0 + \varepsilon_h x)|v_h|^2\, dx \\
&\leq \left(\frac{1}{2} - \frac{1}{p+1}\right) \liminf_{h\to\infty} \int_{\mathbb{R}^3} \left| \left(\frac{1}{i}\nabla - A(z_0 + \varepsilon_h x)\right) v_h \right|^2 + V(z_0 + \varepsilon_h x)|v_h|^2\, dx \\
&\leq \liminf_{h\to\infty} \varepsilon_h^{-3} J_{\varepsilon_h}(u_{\varepsilon_h}) = \Sigma_r(z_0),
\end{aligned}$$

where we have used the diamagnetic inequality (2.2) with $\varepsilon = 1$. Hence we get $\vartheta = 1$, which gives at once $\tilde{v} \in \mathcal{N}_{z_0}$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 + V(z_0)|\tilde{v}|^2 dx &\leq \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} \liminf_{h\to\infty} \varepsilon_h^{-3} J_{\varepsilon_h}(u_{\varepsilon_h}) \\
&= \left(\frac{1}{2} - \frac{1}{p+1}\right)^{-1} \Sigma_r(z_0) \leq \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 + V(z_0)|\tilde{v}|^2 dx.
\end{aligned}$$

This implies that $|v_h| \to \tilde{v}$ strongly in $H^1(\mathbb{R}^3, \mathbb{R})$. Repeating the arguments in the proof of [19, Lemma 5] we conclude that $z_0 \in \mathcal{C}$ (the concentration occurs exponentially fast, see Step II of the proof of Theorem 3.1). This proves that $\mathcal{E} \subset \mathcal{C}$. The converse inclusion follows by the uniqueness of solutions (up to translations) to problem (1.3). Indeed, if $z_0 \in \mathcal{C}$, the sequence $\varepsilon_h^{-3} J_{\varepsilon_h}(u_{\varepsilon_h})$ converges to $J_{z_0}(v_0)$ being v_0 an element of the family

$$\{e^{i\Upsilon_{z_0}(x) + i\omega} \phi_0(x)\}_{\omega \in \mathbb{R}},$$

where ϕ_0 is the unique solution to (1.3) up to translations (cf. [19, Lemma 7]). In particular, there holds $J_{z_0}(v_0) = I_{z_0}(\phi_0) = \Sigma_r(z_0)$, that is $z_0 \in \mathcal{E}$, concluding the proof. For similar considerations in the case $A = 0$, see e.g. [18, Lemma 4.2]. $\square$

We are naturally led to consider the following question (see also Remark 3.1).

Question 2.1. When $f(u)$ does *not* satisfy (2.6), is it still true that $\mathcal{E} = \mathcal{C}$?

3. The Main Result

For every $p \in (1,5)$, let us set

$$\mathfrak{S}_p := \big\{ z \in \mathbb{R}^3 \colon (5-p)K(z)\nabla V(z) = 4V(z)\nabla K(z) \big\}.$$

We now come to the main result of the paper.

Theorem 3.1. *Assume that there exist $C \geq 0$ and $\gamma > 0$ such that, for $|x|$ large,*

$$|A'(x)| \leq Ce^{\gamma|x|}, \quad |\nabla V(x)| \leq Ce^{\gamma|x|}, \quad |\nabla K(x)| \leq Ce^{\gamma|x|}. \tag{3.1}$$

Let $(u_{\varepsilon_h}) \subset \mathcal{H}^{\varepsilon_h}_{A,V}$ be a sequence of bound-state solutions to (S_ε). Then,

$$\mathcal{C} \subset \mathfrak{S}^* \quad \text{and} \quad \mathcal{E} \subset \mathfrak{S}.$$

If in addition f satisfies (2.6), then we have

$$\mathcal{C} = \mathcal{E} \subset \mathfrak{S} = \mathfrak{S}_p.$$

Proof. Let $z_0 \in \mathcal{C}$ and set $v_h(x) = u_{\varepsilon_h}(z_0 + \varepsilon_h x)$ for every $h \geq 1$ and $x \in \mathbb{R}^3$. Then, the sequence (v_h) satisfies the rescaled equation

$$\begin{aligned} -\Delta v_h - \frac{2}{i}\,\langle A(z_0+\varepsilon_h x)\,|\,\nabla v_h\rangle - \frac{\varepsilon_h}{i}\operatorname{div} A(z_0+\varepsilon_h x)v_h \\ + |A(z_0+\varepsilon_h x)|^2 v_h + V(z_0+\varepsilon_h x)v_h = K(z_0+\varepsilon_h x)f(|v_h|^2)v_h. \end{aligned} \tag{3.2}$$

We shall divide the proof into five steps.

Step I. Up to a subsequence, (v_h) converges in some Hölder space $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ to the function $v_0(x) = e^{i\Upsilon_{z_0}(x)}U_{z_0}(x)$, where $U_{z_0} \colon \mathbb{R}^3 \to \mathbb{C}$ is a solution to the equation

$$-\Delta U_{z_0} + V(z_0)U_{z_0} = K(z_0)f(|U_{z_0}|^2)U_{z_0}. \tag{3.3}$$

By the assumption on (u_{ε_h}), the sequence (v_h) is bounded in $\mathcal{H}^1_{A,V}$, and the diamagnetic inequality (2.2) immediately implies that $(|v_h|)$ is bounded in $H^1(\mathbb{R}^3,\mathbb{R})$. Therefore, up to a subsequence, it converges weakly in $H^1(\mathbb{R}^3,\mathbb{R})$ and locally strongly in any $L^q(\mathbb{R}^3,\mathbb{R})$ with $q < 6$ towards a positive function v_*. Moreover, for each compact subset $\Lambda \subset \mathbb{R}^3$, by the continuity of A, (v_h) is also bounded in $H^1(\Lambda,\mathbb{C})$. We may now use the subsolution estimate (see e.g. [16, Theorem 8.17]) to get that (v_h) is also bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^3)$ and hence in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$, via Schauder' estimates. By combining this fact with the results of [20], up to a subsequence, v_h converges to v_0 in $C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^3)$ and furthermore $v_0 \not\equiv 0$, since $|v_h(0)| = |u_{\varepsilon_h}(z_0)| \geq \varrho > 0$. By continuity, the limit v_0 satisfies the limiting equation

$$-\Delta v_0 - \frac{2}{i}\langle A(z_0) \,|\, \nabla v_0\rangle + |A(z_0)|^2 v_0 + V(z_0)v_0 = K(z_0)f(|v_0|^2)v_0. \tag{3.4}$$

If we define $U_{z_0}\colon x \in \mathbb{R}^3 \mapsto e^{-i\Upsilon_{z_0}(x)}v_0(x)$, then U_{z_0} satisfies (3.3).

Step II. There exist two positive constants R_* and C_* such that

$$|v_h(x)| \leq C_* e^{-\sqrt{\frac{V_0}{2}}|x|}, \quad \text{for every } |x| \geq R_* \text{ and } h \geq 1, \tag{3.5}$$

where V_0 is defined in (2.1). Since $z_0 \in \mathcal{C}$, we have $v_h(x) \to 0$ as $|x| \to \infty$, uniformly with respect to $h \geq 1$. Hence, for any $\eta > 0$, we can find a radius $R_\eta > 0$ such that $|v_h(x)| < \eta$ whenever $|x| > R_\eta$ and $h \geq 1$. Therefore, exploiting again Kato's inequality

$$\Delta|v_h| \geq \Re(\bar{v}_h|v_h|^{-1}(\nabla - iA)^2 v_h) \quad \text{(in distributional sense)},$$

and taking into account that f is increasing, there holds

$$\Delta|v_h| \geq V(z_0 + \varepsilon_h x)|v_h| - K(z_0+\varepsilon_h x)f(|v_h|^2)|v_h| \geq [V_0 - K_0 f(\eta^2)]|v_h|$$

in the sense of distributions on $\{|x| > R_\eta\}$, where $K_0 > 0$ is as in (2.1). Let Γ_0 be a fundamental solution for $-\Delta + c_\eta$, where $c_\eta = V_0 - K_0 f(\eta^2)$. We can choose Γ_0 so that $|v_h(x)| \leq [V_0 - K_0 f(\eta^2)]\Gamma_0(x)$ holds for $|x| = R_\eta$. Then, if $w = |v_h| - [V_0 - K_0 f(\eta^2)]\Gamma_0$, there holds

$$\begin{aligned}\Delta w &= \Delta|v_h| - [V_0 - K_0 f(\eta^2)]\Delta\Gamma_0 \\ &\geq [V_0 - K_0 f(\eta^2)]|v_h| - [V_0 - K_0 f(\eta^2)]^2\Gamma_0 \\ &= [V_0 - K_0 f(\eta^2)]w\end{aligned}$$

in distributional sense over $\{|x| > R_\eta\}$. Then, by the maximum principle, $w(x) \leq 0$ for every $|x| \geq R_\eta$. Since, as known, Γ_0 decays exponentially at the rate $\sqrt{c_\eta}$, fixing $\eta = \eta_*$ so small that $f(\eta_*^2) \leq V_0/2K_0$, we can find constants $R_* > 0$ and $c > 0$ such that $\Gamma_0(x) \leq c\exp\{-\sqrt{V_0/2}|x|\}$ for $|x| \geq R_*$, which yields the desired conclusion.

Step III. For every $h \geq 1$, the following identity holds

$$\begin{aligned}\int_{\mathbb{R}^3}\Bigg[&\left\langle \frac{\partial A}{\partial x_k}(z_0+\varepsilon_h x)\,\middle|\, A(z_0+\varepsilon_h x)\right\rangle |v_h|^2 - \Re\left\langle \frac{1}{i}\nabla v_h \,\middle|\, \frac{\partial A}{\partial x_k}(z_0+\varepsilon_h x)\bar{v}_h\right\rangle \\ &+ \frac{\partial V}{\partial x_k}(z_0+\varepsilon_h x)\frac{|v_h|^2}{2} - \frac{\partial K}{\partial x_k}(z_0+\varepsilon_h x)F(|v_h|^2)\Bigg]\,dx = 0.\end{aligned} \tag{3.6}$$

Rigorously, we cannot directly apply the Pucci–Serrin variational identity [25], since the solutions to Eq. (3.2) are complex-valued. For we are not aware of any explicit reference to cite for the identity we need, we will derive (3.6) directly (see also [9]).

Throughout the rest of this step only, we use the less cumbersome notation $x \cdot y$ in place of $\langle x \mid y \rangle$ to indicate the standard scalar product in $\mathbb{R}^3$.

First of all, let us observe that, for every $h \geq 1$,

$$|\nabla v_h| \leq |D^1 v_h| + |A(z_0 + \varepsilon_h x)||v_h|.$$

Hence, taking into account Step II and the bounds (3.1) and (2.3), we get

$$\begin{aligned} \|\nabla v_h\|_{L^2(\mathbb{R}^3)} &\leq \|D^1 v_h\|_{L^2(\mathbb{R}^3)} + \|A(z_0 + \varepsilon_h x) v_h\|_{L^2(\mathbb{R}^3)} \\ &\leq \|D^1 v_h\|_{L^2(\mathbb{R}^3)} + |A(z_0)| \|v_h\|_{L^2(\mathbb{R}^3)} + c\|e^{\gamma \varepsilon_h |x|}|x| v_h\|_{L^2(\mathbb{R}^3)} \\ &\leq c, \end{aligned} \tag{3.7}$$

for all $h \geq 1$ and some $c > 0$. Let $\delta > 0$ and consider the cut-off function $\psi_\delta = \psi(\delta x)$, where $\psi \in C_c^1(\mathbb{R}^3)$ is such that $\psi(x) = 1$ for $|x| \leq 1$ and $\psi(x) = 0$ for $|x| \geq 2$. If $\boldsymbol{e}_k$ denotes the kth vector of the canonical base in $\mathbb{R}^3$, we test Eq. (3.2) with the function $\psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}$ and we take the real part. Firstly, we have

$$\Re \int_{\mathbb{R}^3} \nabla v_h \cdot \nabla[\psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}]\, dx = \Re \int_{\mathbb{R}^3} \nabla v_h \cdot \nabla \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}\, dx - \int_{\mathbb{R}^3} \nabla \psi_\delta \cdot \boldsymbol{e}_k \frac{|\nabla v_h|^2}{2}\, dx.$$

As a consequence, by virtue of (3.7), the Dominate Convergence Theorem yields

$$\lim_{\delta \to 0} \Re \int_{\mathbb{R}^3} \nabla v_h \cdot \nabla[\psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}]\, dx = 0.$$

Now, we have

$$\begin{aligned} &\Re \int_{\mathbb{R}^3} K(z_0 + \varepsilon_h x) f(|v_h|^2) v_h \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}\, dx \\ &\quad = \Re \int_{\mathbb{R}^3} K(z_0 + \varepsilon_h x) \psi_\delta \boldsymbol{e}_k \cdot \nabla F(|v_h|^2)\, dx \\ &\quad = -\varepsilon_h \int_{\mathbb{R}^3} \frac{\partial K}{\partial x_k}(z_0 + \varepsilon_h x) \psi_\delta F(|v_h|^2)\, dx - \int_{\mathbb{R}^3} K(z_0 + \varepsilon_h x) \frac{\partial \psi_\delta}{\partial x_k} F(|v_h|^2)\, dx. \end{aligned}$$

Hence, in light of (3.1), (3.5) and (3.7), by the Dominate Convergence theorem we have

$$\lim_{\delta \to 0} \Re \int_{\mathbb{R}^3} K(z_0 + \varepsilon_h x) f(|v_h|^2) v_h \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h} dx = -\varepsilon_h \int_{\mathbb{R}^3} \frac{\partial K}{\partial x_k}(z_0 + \varepsilon_h x) F(|v_h|^2)\, dx.$$

In a similar fashion, there hold

$$\begin{aligned} &\lim_{\delta \to 0} \Re \int_{\mathbb{R}^3} V(z_0 + \varepsilon_h x) v_h \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}\, dx = -\varepsilon_h \int_{\mathbb{R}^3} \frac{\partial V}{\partial x_k}(z_0 + \varepsilon_h x) \frac{|v_h|^2}{2}\, dx, \\ &\lim_{\delta \to 0} \Re \int_{\mathbb{R}^3} |A(z_0 + \varepsilon_h x)|^2 v_h \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}\, dx \\ &\quad = -\varepsilon_h \int_{\mathbb{R}^3} A(z_0 + \varepsilon_h x) \cdot \frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x) |v_h|^2\, dx. \end{aligned}$$

Finally, we have

$$J(\delta) = \Re \int_{\mathbb{R}^3} \frac{2}{i} A(z_0 + \varepsilon_h x) \cdot \nabla v_h \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h}\, dx = J_1(\delta) + J_2(\delta) + J_3(\delta),$$

where we have set

$$J_1(\delta) = -\varepsilon_h \Re \sum_{m=1}^{3} \int_{\mathbb{R}^3} \frac{2}{i} \frac{\partial A_m}{\partial x_k}(z_0 + \varepsilon_h x)\psi_\delta \frac{\partial v_h}{\partial x_m} \bar{v}_h \, dx,$$

$$J_2(\delta) = -\Re \sum_{m=1}^{3} \int_{\mathbb{R}^3} \frac{2}{i} A_m(z_0 + \varepsilon_h x) \frac{\partial \psi_\delta}{\partial x_k} \frac{\partial v_h}{\partial x_m} \bar{v}_h \, dx,$$

$$J_3(\delta) = -\Re \sum_{m=1}^{3} \int_{\mathbb{R}^3} \frac{2}{i} A_m(z_0 + \varepsilon_h x)\psi_\delta \frac{\partial^2 v_h}{\partial x_k \partial x_m} \bar{v}_h \, dx.$$

After a few computations, one shows that $J_2(\delta) \to 0$ as $\delta \to 0$ and

$$J_3(\delta) = -\Re \int_{\mathbb{R}^3} \frac{2\varepsilon_h}{i} \mathrm{div} A(z_0 + \varepsilon_h x) v_h \psi_\delta \boldsymbol{e}_k \cdot \overline{\nabla v_h} \, dx - J(\delta) + \Theta(\delta),$$

with $\Theta(\delta) \to 0$ as $\delta \to 0$. Furthermore, again by (3.1), (3.5) and (3.7)

$$\lim_{\delta \to 0} J_1(\delta) = -\varepsilon_h \Re \int_{\mathbb{R}^3} \frac{2}{i} \nabla v_h \cdot \frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x)\bar{v}_h \, dx.$$

Therefore, we obtain

$$\begin{aligned}\lim_{\delta \to 0} J(\delta) = &-\Re \int_{\mathbb{R}^3} \frac{\varepsilon_h}{i} \mathrm{div} A(z_0 + \varepsilon_h x) v_h \boldsymbol{e}_k \cdot \overline{\nabla v_h} \, dx \\ &- \varepsilon_h \Re \int_{\mathbb{R}^3} \frac{1}{i} \nabla v_h \cdot \frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x)\bar{v}_h \, dx.\end{aligned}$$

Adding the above identities immediately yields (3.6).

Step IV. We apply the Dominate Convergence Theorem to take the limit as $h \to \infty$ into identity (3.6). The only troublesome term is

$$\Re \left\langle \frac{1}{i} \nabla v_h \middle| \frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x)\bar{v}_h \right\rangle,$$

since we apparently have no control on the decay of ∇v_h. Taking into account (3.7) and recalling that $\nabla v_h(x) \to \nabla v_0(x)$ for all $x \in \mathbb{R}^3$, up to a subsequence, we have

$$\nabla v_h \rightharpoonup \nabla v_0, \quad \text{weakly in } L^2(\mathbb{R}^3). \tag{3.8}$$

On the other hand, by virtue of Step II, there exist $R_* > 0$ and $c > 0$ such that

$$\left| \frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x)\bar{v}_h \right| \leq c e^{-\left(\sqrt{\frac{V_0}{2}} - \gamma \varepsilon_h\right)|x|}, \quad \text{for every } |x| \geq R_*.$$

Consequently, since $\bar{v}_h(x) \to \bar{v}_0(x)$ for all $x \in \mathbb{R}^3$ and $A \in C^1(\mathbb{R}^3)$, there holds

$$\frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x)\bar{v}_h \to \frac{\partial A}{\partial x_k}(z_0)\bar{v}_0, \quad \text{strongly in } L^2(\mathbb{R}^3). \tag{3.9}$$

Thus, by combining (3.8) and (3.9), for each k, we immediately get

$$\lim_{h \to \infty} \int_{\mathbb{R}^3} \Re \left\langle \frac{1}{i} \nabla v_h \middle| \frac{\partial A}{\partial x_k}(z_0 + \varepsilon_h x)\bar{v}_h \right\rangle dx = \int_{\mathbb{R}^3} \Re \left\langle \frac{1}{i} \nabla v_0 \middle| \frac{\partial A}{\partial x_k}(z_0)\bar{v}_0 \right\rangle dx.$$

Since similar considerations apply to the other terms that appear in (3.6), we can therefore pass to the limit as $h \to \infty$, to find, for each k,

$$\int_{\mathbb{R}^3} \left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| A(z_0) \right\rangle |v_0|^2 - \Re\left\langle \frac{1}{i}\nabla v_0 \middle| \frac{\partial A}{\partial x_k}(z_0)\bar{v}_0 \right\rangle dx \\ + \frac{\partial V}{\partial x_k}(z_0) \int_{\mathbb{R}^3} \frac{|v_0|^2}{2}\, dx - \frac{\partial K}{\partial x_k}(z_0) \int_{\mathbb{R}^3} F(|v_0|^2)\, dx = 0. \tag{3.10}$$

Now, as proved in Step I, v_0 can be represented as $v_0(x) = e^{i\Upsilon_{z_0}(x)} U_{z_0}(x)$ where $U_{z_0}\colon \mathbb{R}^3 \to \mathbb{C}$ solves (3.3). Taking into account that

$$\frac{1}{i}\nabla v_0(x) = e^{i\Upsilon_{z_0}(x)} A(z_0) U_{z_0}(x) - i e^{i\Upsilon_{z_0}(x)} \nabla U_{z_0}(x),$$

for every $x \in \mathbb{R}^3$ we obtain

$$\begin{aligned}
&\left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| A(z_0) \right\rangle |v_0(x)|^2 - \Re\left\langle \frac{1}{i}\nabla v_0(x) \middle| \frac{\partial A}{\partial x_k}(z_0)\bar{v}_0(x) \right\rangle \\
&\quad = \left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| A(z_0) \right\rangle |U_{z_0}(x)|^2 \\
&\qquad - \Re\left\langle e^{i\Upsilon_{z_0}(x)} A(z_0) U_{z_0}(x) - i e^{i\Upsilon_{z_0}(x)} \nabla U_{z_0}(x) \middle| \frac{\partial A}{\partial x_k}(z_0) e^{-i\Upsilon_{z_0}(x)} \bar{U}_{z_0}(x) \right\rangle \\
&\quad = \left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| A(z_0) \right\rangle |U_{z_0}(x)|^2 \\
&\qquad - \Re\left(\left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| A(z_0) \right\rangle |U_{z_0}(x)|^2 - i \left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| \nabla U_{z_0}(x) \right\rangle \bar{U}_{z_0}(x) \right) \\
&\quad = \left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| \Re(i\bar{U}_{z_0}(x) \nabla U_{z_0}(x)) \right\rangle.
\end{aligned}$$

Hence, equation (3.10) can be rephrased as

$$\left\langle \frac{\partial A}{\partial x_k}(z_0) \middle| \int_{\mathbb{R}^3} \Re(i\bar{U}_{z_0} \nabla U_{z_0})\, dx \right\rangle \\ + \frac{\partial V}{\partial x_k}(z_0) \int_{\mathbb{R}^3} \frac{|U_{z_0}|^2}{2}\, dx - \frac{\partial K}{\partial x_k}(z_0) \int_{\mathbb{R}^3} F(|U_{z_0}|^2)\, dx = 0,$$

for every $k = 1, 2, 3$, namely,

$$\left\langle \frac{\partial A}{\partial w}(z_0) \middle| \int_{\mathbb{R}^3} \Re(i\bar{U}_{z_0} \nabla U_{z_0})\, dx \right\rangle \\ + \frac{\partial V}{\partial w}(z_0) \int_{\mathbb{R}^3} \frac{|U_{z_0}|^2}{2}\, dx - \frac{\partial K}{\partial w}(z_0) \int_{\mathbb{R}^3} F(|U_{z_0}|^2)\, dx = 0, \tag{3.11}$$

for every $w \in \mathbb{R}^3$.

Step V. In this final step, we prove the desired inclusions stated by the theorem. As a consequence of identity (3.11), in light of the definition of $\Gamma^{\pm}(z_0; w)$, we immediately deduce that $z_0 \in \mathfrak{S}^*$, thus proving that $\mathcal{C} \subset \mathfrak{S}^*$. Let us now assume that $z_0 \in \mathcal{E}$. Then $J_{z_0}(v_0) = \Sigma_c(z_0) = \Sigma_r(z_0)$, and by virtue of (iii) of Lemma 2.2, we have $U_{z_0}(x) = e^{i\omega} u_{z_0}(x)$ for some $\omega \in \mathbb{R}$, where u_{z_0} is a real-valued least energy solution to (1.3). Moreover, by (ii) of Lemma 2.2, we have

$$\Re\left(i\bar{U}_{z_0}(x)\nabla U_{z_0}(x)\right) = 0, \quad \text{for a.e. } x \in \mathbb{R}^3.$$

Then, in light of Lemma 2.1 and (3.11), we obtain

$$\begin{aligned}
\left(\frac{\partial \Sigma_r}{\partial w}\right)^-(z_0) &= \sup_{u \in S_r(z_0)} \left[\frac{\partial V}{\partial w}(z_0)\int_{\mathbb{R}^3} \frac{|u|^2}{2}\,dx + \frac{\partial K}{\partial w}(z_0)\int_{\mathbb{R}^3} F(|u|^2)\,dx\right] \\
&= \sup_{\substack{U = e^{i\omega} u \\ u \in S_r(z_0)}} \left[\frac{\partial V}{\partial w}(z_0)\int_{\mathbb{R}^3} \frac{|U|^2}{2}\,dx + \frac{\partial K}{\partial w}(z_0)\int_{\mathbb{R}^3} F(|U|^2)\,dx\right] \\
&\geq \frac{\partial V}{\partial w}(z_0)\int_{\mathbb{R}^3} \frac{|U_{z_0}|^2}{2}\,dx + \frac{\partial K}{\partial w}(z_0)\int_{\mathbb{R}^3} F(|U_{z_0}|^2)\,dx = 0,
\end{aligned}$$

for every $w \in \mathbb{R}^3$. In a similar fashion, there holds

$$\left(\frac{\partial \Sigma_r}{\partial w}\right)^+(z_0) \leq 0,$$

for every $w \in \mathbb{R}^3$. In particular, by the definition of $(-\Sigma_r)^0(z_0; w)$, we get

$$(-\Sigma_r)^0(z_0; w) \geq \left(\frac{\partial(-\Sigma_r)}{\partial w}\right)^+(z_0) \geq 0,$$

for $w \in \mathbb{R}^3$. Hence $0 \in \partial_C(-\Sigma_r)(z_0)$, which, in light of Proposition 2.1, yields $z_0 \in \mathfrak{S}$. Finally, if $f(u)$ satisfies (2.6), problem (1.3) admits a unique (up to translations) real-valued solution ϕ_0 (see [5]). Taking into account Lemma 2.2, there exists $\omega \in \mathbb{R}$ such that $v_0 = e^{i\Upsilon_{z_0}(x) + i\omega}\phi_0(x)$. Then, if $z_0 \in \mathcal{C} = \mathcal{E}$ (see Proposition 2.1), we have $S_r(z_0) = \{\phi_0\}$, Σ_r admits all the directional derivatives and, by the above inequalities,

$$\left(\frac{\partial \Sigma_r}{\partial w}\right)^{\pm}(z_0) = \frac{\partial \Sigma_r}{\partial w}(z_0) = 0,$$

for $w \in \mathbb{R}^3$. Since up to a multiplicative constant Σ_r writes down explicitly as (1.2), the last assertion readily follows by a direct computation. □

In light of identity (3.11), we also have the following

Corollary 3.1. *Under the assumptions of Theorem* 3.1, *for every* $z_0 \in \mathcal{C}$, *there exist constants* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ (*possibly zero*) *and* $\gamma_1, \gamma_2 \in \mathbb{R}\backslash\{0\}$ *such that*

$$\sum_{j=1}^{3} \lambda_j \nabla A_j(z_0) + \gamma_1 \nabla V(z_0) + \gamma_2 \nabla K(z_0) = 0. \tag{3.12}$$

Hence, in general, the location of concentration points might depend also on the (fixed) external electromagnetic potential A. *If* $\mathfrak{S}^* = \emptyset$, *then* (S_ε) *does not admit any sequence of bound-state solutions concentrating somewhere pointwise.*

Corollary 3.2. *The location of energy-concentration points of a sequence of bound-state solutions to problem* (S_ε) *is independent of the external electromagnetic field* B (*and there holds* $\lambda_j = 0$ *for all* $j = 1, 2, 3$ *in* (3.12)). *If* $\mathfrak{S} = \emptyset$, *then* (S_ε) *does not admit any sequence of bound-state solutions concentrating somewhere energetically.*

Remark 3.1. Despite the fact that both pointwise and energy concentration are gauge invariant, the necessary condition (3.12) is not, in general, unless $\lambda_j = 0$ for all $j = 1, 2, 3$. Hence, it seems natural to conjecture that the answer to Question 2.1 is always affirmative.

Corollary 3.3. *Assume that* $f(u)$ *is such that, for every* $z \in \mathbb{R}^3$, *problem* (1.3) *admits a unique positive radial solution, up to translations. Then, if* z *is an energy-concentration point it is a classical critical point of* Σ_r.

We refer the reader to [5, Theorems 2.5 and 4.2] for some results ensuring uniqueness for (1.3) under some additional hypothesis on $f(u)$.

We finish the paper with a simple but interesting property of the family $\{\mathfrak{S}_p\}_{p\in(1,5)}$.

Observation 3.1. Assume that $f(u)$ satisfies (2.6) and that

$$\limsup_{|x|\to\infty} \frac{|\nabla V(x)|}{V(x)} < \infty \quad \text{and} \quad \liminf_{|x|\to\infty} |\nabla K(x)| > 0.$$

We denote by $\mathrm{Crit}(K)$ the set of critical points of K, which is a compact set in light of the above assumption. Then, it is a simple task to check that

$$\lim_{p\to 5^-} \mathrm{dist}_{\mathbb{R}^3}(\mathfrak{S}_p, \mathrm{Crit}(K)) = 0,$$

that is, if p is close to the critical exponent 5, the spikes locate close to $\mathrm{Crit}(K)$.

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