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Symmetry in variational principles and applications

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Abstract

We formulate symmetric versions of classical variational principles. Within the framework of nonsmooth critical point theory, we detect Palais–Smale sequences with additional second-order and symmetry information. We discuss applications to partial differential equations, fixed point theory and geometric analysis.

1. *Introduction*

One of the most powerful contributions of the last decades in calculus of variations and nonlinear analysis is surely given by Ekeland's variational principle for lower semi-continuous functionals on complete metric spaces [**15**, **16**], arisen in the context of convex analysis. We refer to [**1**, **4**, **16**, **19**, **21**] where a multitude of applications in different fields of analysis is carefully discussed. In a recent note [**27**], Squassina has proved a version of the principle in Banach spaces that provides *almost symmetric* almost critical points, provided that the functional satisfies a rather mild symmetry condition. Roughly speaking, if $(X, \|\cdot\|)$ is a Banach space that is continuously embedded into a space $(V, \|\cdot\|_V)$ with suitable properties and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous bounded-below functional that does not increase by polarization, then, for all $\varepsilon > 0$, there is $u_\varepsilon \in X$ with

$$\|u_\varepsilon - u_\varepsilon^*\|_V < \varepsilon, \quad f(u_\varepsilon) \leqslant \inf f + \varepsilon^2, \quad f(\xi) \geqslant f(u_\varepsilon) - \varepsilon\|\xi - u_\varepsilon\| \ \ \forall \xi \in X,$$

where the symmetrization $*$ is defined in an abstract framework, which reduces to the classical notions in concrete functional spaces, such as in $L^p(\Omega)$ and in $W_0^{1,p}(\Omega)$ spaces, Ω being either a ball in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$. Possessing almost symmetric points is very useful in applications not only to find symmetric cluster points, but also to facilitate the strong convergence of the sequence (u_ε) via suitable compact embeddings enjoyed by spaces X_* of symmetric functions of X (see [**22**, **28**, **31**]). The aim of the present manuscript is to give a rather complete range of abstract results and in this direction to furnish also applications to calculus of variations, fixed point theory and geometry of Banach spaces.

The plan of the paper is as follows. In Section 2, we state symmetric versions of Ekeland [**15**], Borwein–Preiss [**3**] and Deville–Godefroy–Zizler [**13**] principles, free or constrained, as well as versions for the Ekeland's principle with weights, in the spirit of Zhong's result [**32**] (see Theorems 2.5, 2.7, 2.8, 2.11–2.13, 2.18 and 2.20). Furthermore, in the framework of the nonsmooth critical point theory developed in [**12**], we detect suitable Palais–Smale (PS) sequences (u_h) with respect to the notion of weak slope whose elements u_h become more and more symmetric, $u_h \sim u_h^*$, as h gets large, and satisfy a second-order information, in terms of a quantity $w \mapsto Q_{u_h}(w)$, introduced in [**2**], that plays the rǒle of the quadratic form $w \mapsto f''(u_h)(w)^2$ for functions of class C^2 (see Theorem 2.28 as well as Corollary 2.30). As pointed out by Lions [**23**], this additional second-order information can be very important

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to prove the strong convergence of $(u_h) \subset X$, in some physically meaningful situations. It would be interesting to obtain results in the same spirit for mountain pass values in place of minimum values, as developed, by Fang and Ghoussoub [**18**] without symmetry information. In Section 2.5, a discussion upon the relationships between symmetry, coercivity and PS sequences is developed while in Section 2.7 an application of the symmetric Ekeland principles to get minimax-type results is outlined. In Section 3, we discuss some possible applications and implications of the abstract machinery formulated in Section 2. First, we find almost symmetric solutions, up to a perturbation, for two classes of nonlinear elliptic partial differential equations (PDEs) associated with suitable energy functionals (see Theorems 3.2 and 3.4). Next, we get a symmetric version of the Caristi [**6**] fixed point theorem and of a theorem due to Clarke [**8**] (see Theorems 3.5 and 3.8) and we obtain some applications in the geometry of Banach spaces, such as symmetric versions of Daneš Drop [**10**] and Flower Petal theorems [**26**] (see Theorems 3.11 and 3.13).

2. *Symmetric variational principles*

Let X, V and W be three real Banach spaces with $X \subseteq V \subseteq W$ and let $S \subseteq X$.

2.1. *Abstract framework*

Following [**30**], consider the following definition.

Definition 2.1. We consider two maps $* : S \to V$, $u \mapsto u^*$, the symmetrization map, and $h : S \times \mathcal{H}_* \to S$, $(u, H) \mapsto u^H$, the polarization map, $\mathcal{H}_*$ being a path-connected topological space. We assume that the following hold:

(i) X is continuously embedded in V; V is continuously embedded in W;
(ii) h is a continuous mapping;
(iii) for each $u \in S$ and $H \in \mathcal{H}_*$ it holds $(u^*)^H = (u^H)^* = u^*$ and $u^{HH} = u^H$;
(iv) there exists $(H_m) \subset \mathcal{H}_*$ such that, for $u \in S$, $u^{H_1 \cdots H_m}$ converges to u^* in V;
(v) for every $u, v \in S$ and $H \in \mathcal{H}_*$ it holds $\|u^H - v^H\|_V \leqslant \|u - v\|_V$.

Moreover, the mappings $h : S \times \mathcal{H}_* \to S$ and $* : S \to V$ can be extended to $h : X \times \mathcal{H}_* \to S$ and $* : X \to V$ by setting $u^H := (\Theta(u))^H$ for every $u \in X$ and $H \in \mathcal{H}_*$ and $u^* := (\Theta(u))^*$ for every $u \in X$, respectively, where $\Theta : (X, \|\cdot\|_V) \to (S, \|\cdot\|_V)$ is a Lipschitz function, of Lipschitz constant $C_\Theta > 0$, such that $\Theta|_S = \mathrm{Id}|_S$.

The previous properties, in particular (iv) and (v), and the definition of Θ easily yield

$$\forall u, v \in X,\ \forall H \in \mathcal{H}_* :\ \|u^H - v^H\|_V \leqslant C_\Theta \|u - v\|_V, \quad \|u^* - v^*\|_V \leqslant C_\Theta \|u - v\|_V. \tag{2.1}$$

For the sake of completeness, we now recall some concrete notions.

2.1.1. *Concrete polarization.* A subset H of $\mathbb{R}^N$ is called a polarizer if it is a closed affine half-space of $\mathbb{R}^N$, namely the set of points x that satisfy $\alpha \cdot x \leqslant \beta$ for some $\alpha \in \mathbb{R}^N$ and $\beta \in \mathbb{R}$ with $|\alpha| = 1$. Given x in $\mathbb{R}^N$ and a polarizer H, the reflection of x with respect to the boundary of H is denoted by x_H. The polarization of a function $u : \mathbb{R}^N \to \mathbb{R}^+$ by a polarizer H is the function $u^H : \mathbb{R}^N \to \mathbb{R}^+$ defined by

$$u^H(x) = \begin{cases} \max\{u(x), u(x_H)\}, & \text{if } x \in H, \\ \min\{u(x), u(x_H)\}, & \text{if } x \in \mathbb{R}^N \setminus H. \end{cases} \tag{2.2}$$

The polarization $C^H \subset \mathbb{R}^N$ of a set $C \subset \mathbb{R}^N$ is defined as the unique set which satisfies $\chi_{C^H} = (\chi_C)^H$, where χ denotes the characteristic function. The polarization u^H of a positive function u defined on $C \subset \mathbb{R}^N$ is the restriction to C^H of the polarization of the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C. The polarization of a function that may change sign is defined by $u^H := |u|^H$ for any given polarizer H.

2.1.2. *Concrete symmetrization.* The Schwarz symmetrization of $C \subset \mathbb{R}^N$ is the unique open ball centred at the origin C^* such that $\mathcal{L}^N(C^*) = \mathcal{L}^N(C)$ ($\mathcal{L}^N$ being the Lebesgue measure on $\mathbb{R}^N$). If the measure of C is zero, then set $C^* = \emptyset$. If the measure of C is not finite, then put $C^* = \mathbb{R}^N$. A measurable function u is admissible for the Schwarz symmetrization if $u \geqslant 0$ and, for all $\varepsilon > 0$, the measure of $\{u > \varepsilon\}$ is finite. The Schwarz symmetrization of an admissible $u : C \to \mathbb{R}^+$ is the unique $u^* : C^* \to \mathbb{R}^+$ such that, for all $t \in \mathbb{R}$, it holds $\{u^* > t\} = \{u > t\}^*$. Considering the extension $\tilde{u} : \mathbb{R}^N \to \mathbb{R}^+$ of u by zero outside C, it is $(\tilde{u})^*|_{\mathbb{R}^N \setminus C^*} = 0$ and $u^* = (\tilde{u})^*|_{C^*}$. Let $\mathcal{H}_* = \{H \in \mathcal{H} : 0 \in H\}$ and let Ω be a ball in $\mathbb{R}^N$ or the whole space $\mathbb{R}^N$. Then $u = u^*$ if and only if $u = u^H$ for every $H \in \mathcal{H}_*$. Set either $X = W_0^{1,p}(\Omega)$, $S = W_0^{1,p}(\Omega, \mathbb{R}^+)$, $V = L^p \cap L^{p^*}(\Omega)$ with $h(u) := u^H$ and $*(u) := u^*$ or $X = S = W_0^{1,p}(\Omega)$, $V = L^p \cap L^{p^*}(\Omega)$ with $h(u) := |u|^H$ and $*(u) := |u|^*$. In the first case $\Theta(u) := |u|$ defines a function from $(X, \|\cdot\|_V)$ to $(S, \|\cdot\|_V)$, Lipschitz of constant $C_\Theta = 1$, allowing to extend the definition of $h, *$ on $X = W_0^{1,p}(\Omega)$ by $h(u) := h(\Theta(u))$ and $*(u) := *(\Theta(u))$. In both cases properties (i)–(v) in Definition 2.1 are satisfied [**30**].

We now recall [**30**, Corollary 3.1] a useful result on the approximation of symmetrizations. The subset S of X in Definition 2.1 is considered as a metric space with the metric d induced by $\|\cdot\|$ on X. We assume that conditions (i)–(v) of Definition 2.1 are satisfied.

PROPOSITION 2.2. *For all $\rho > 0$ there exists a continuous mapping $\mathbb{T}_\rho : S \to S$ such that $\mathbb{T}_\rho u$ is built via iterated polarizations and $\|\mathbb{T}_\rho u - u^*\|_V < \rho$ for all $u \in S$.*

REMARK 2.3. If S is the set involved in Definition 2.1, then assume that

$$S' \subseteq S, \quad h(S' \times \mathcal{H}_*) \subseteq S', \quad *(S') \subseteq V.$$

Then $(S', X, V, h, *)$ satisfies conditions (i)–(v) of Definition 2.1 and Proposition 2.2 holds for S' in place of S. If $u \in X$, then one still defines $u^H := (\Theta(u))^H$ and $u^* := (\Theta(u))^*$ for all $u \in X$. Note that $\Theta(u) = u$ for all $u \in S'$, since $S' \subseteq S$ and $\Theta|_S = \mathrm{Id}|_S$.

2.2. *Classical principles*

In the following, we recall a particular form, suitable for our purposes, of Borwein–Preiss's smooth variational principle [**3**] for reflexive Banach spaces endowed with a Kadec renorm (cf. [**3**, Theorems 2.6 and 5.2, and Formula 5.4]). We say that X is endowed with a Kadec renorm $\|\cdot\|$, if the weak and norm topologies agree on the unit sphere of X. Such a norm indeed exists if X is reflexive [**14**].

THEOREM 2.4 (Borwein–Preiss's principle). *Assume that X is a reflexive Banach space, endowed with any Kadec norm $\|\cdot\|$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper bounded-below lower semi-continuous functional. Let $u \in X$, $\rho > 0$, $\sigma > 0$ and $p \geqslant 1$ be such that*

$$f(u) < \inf f + \sigma \rho^p.$$

Then there exist $v \in X$ and $\eta \in X$ such that

(a) $\|v - u\| < \rho$;
(b) $\|\eta - u\| \leqslant \rho$;

(c) $f(v) < \inf f + \sigma\rho^p$;
(d) $f(w) \geqslant f(v) + \sigma(\|v-\eta\|^p - \|w-\eta\|^p)$ *for all* $w \in X$.

The following is a symmetric version of Borwein–Preiss's smooth variational principle.

Theorem 2.5 (Symmetric Borwein–Preiss's principle). *Assume that X is a reflexive Banach space, endowed with any Kadec norm $\|\cdot\|$. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper bounded-below lower semi-continuous functional such that*

$$f(u^H) \leqslant f(u), \quad \text{for all } u \in S \text{ and } H \in \mathcal{H}_*. \tag{2.3}$$

Let $u \in S$, $\rho > 0$, $\sigma > 0$ and $p \geqslant 1$ be such that

$$f(u) < \inf f + \sigma\rho^p. \tag{2.4}$$

Then there exist $v \in X$ and $\eta \in X$ such that

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)\rho$;
(b) $\|v - u\| < \rho + \|\mathbb{T}_\rho u - u\|$;
(c) $\|\eta - u\| \leqslant \rho + \|\mathbb{T}_\rho u - u\|$;
(d) $f(v) < \inf f + \sigma\rho^p$;
(e) $f(w) \geqslant f(v) + \sigma(\|v-\eta\|^p - \|w-\eta\|^p)$ *for all* $w \in X$.

Here $K > 0$ denotes the continuity constant for the injection $X \hookrightarrow V$.

Proof. Let $u \in S$, $\rho > 0$, $\sigma > 0$ and $p \geqslant 1$ be such that $f(u) < \inf f + \sigma\rho^p$. If $\mathbb{T}_\rho : S \to S$ is the mapping of Proposition 2.2, then we set $\tilde{u} := \mathbb{T}_\rho u \in S$. Then, by construction, we have $\|\tilde{u} - u^*\|_V < \rho$ and, in light of (2.3) and the property that $\tilde{u}$ is built from u through iterated polarizations, we obtain $f(\tilde{u}) < \inf f + \sigma\rho^p$. By Theorem 2.4 there exist elements $v \in X$ and $\eta \in X$ with $\|\eta - \tilde{u}\| \leqslant \rho$, such that $f(v) < \inf f + \sigma\rho^p$, $\|v - \tilde{u}\| < \rho$ and

$$f(w) \geqslant f(v) + \sigma(\|v-\eta\|^p - \|w-\eta\|^p), \quad \text{for all } w \in X.$$

Hence, (d) and (e) hold true. Taking into account, the second inequality in (2.1), if $K > 0$ denotes the continuity constant of the injection $X \hookrightarrow V$, then we obtain

$$\|v - v^*\|_V \leqslant K(C_\Theta + 1)\|v - \tilde{u}\| + \|\tilde{u} - u^*\|_V < (K(C_\Theta + 1) + 1)\rho, \tag{2.5}$$

where we used the fact that $u^* = \tilde{u}^*$ in light of (iii) of the abstract framework and, again, by the way $\tilde{u}$ is built from u. Then (a) holds true. Also (b) follows from

$$\|v - u\| \leqslant \|v - \tilde{u}\| + \|\tilde{u} - u\| < \rho + \|\mathbb{T}_\rho u - u\|. \tag{2.6}$$

Finally, (c) holds by virtue of $\|\eta - u\| \leqslant \|\eta - \tilde{u}\| + \|\tilde{u} - u\| \leqslant \rho + \|\mathbb{T}_\rho u - u\|$. $\square$

Remark 2.6. If $u \in S$ in (2.4) is such that $u^H = u$ for all $H \in \mathcal{H}_*$ (which is the case, for instance, if $u^* = u$ and $*$ denotes the usual Schwarz symmetrization in the space of nonnegative vanishing measurable real functions), then by construction $\mathbb{T}_\rho u = u$ for every $\rho > 0$ and conclusions (b)–(c) of Theorem 2.5 improve to

$$\|v - u\| < \rho \quad \text{and} \quad \|\eta - u\| \leqslant \rho. \tag{2.7}$$

Hence, starting with a minimization sequence made of symmetric functions yields a new minimization sequence satisfying (a)–(e) and full smallness controls (b)–(c) of Theorem 2.5. In many concrete cases (although there are some exceptions, as pointed out in [**30**]), if a functional does not increase under polarization, namely, condition (2.3) holds, then it is also nonincreasing under symmetrization, namely

$$f(u^*) \leqslant f(u), \quad \text{for all } u \in S.$$

In these cases, starting from an arbitrary minimization sequence $(u_h) \subset S$, first one can consider the new symmetric minimization sequence $(u_h^*) \subset S$, which already admits a behaviour nicer than that of (u_h), and then apply the variational principle to it, finding a further minimization sequence $(v_h) \subset X$ with even nicer additional properties.

In the abstract framework of Definition 2.1, using Ekeland's principle in complete metric spaces, we can derive the following result.

THEOREM 2.7 (Symmetric Ekeland's principle, I). *Let $S \subset X$ be as in Definition 2.1 and let S' be a closed subset of S satisfying the properties stated in Remark 2.3. Assume that $f : S' \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semi-continuous functional bounded from below such that (2.3) holds (on S'). Then, for all $\rho > 0$ and $\sigma > 0$, there exists $v \in S'$ such that*

(a) $\|v - v^*\|_V < (2K + 1)\rho$;
(b) $f(w) \geqslant f(v) - \sigma\|w - v\|$ *for all* $w \in S'$.

In addition, one can assume that $f(v) \leqslant f(u)$ and $\|v - u\| \leqslant \rho + \|\mathbb{T}_\rho u - u\|$, where $u \in S'$ is some element that satisfies $f(u) \leqslant \inf f + \sigma\rho$.

Proof. As S' is a closed subset of the Banach space X, (S', d) is a complete metric space, where $d(u, v) = \|u - v\|$. Given $\rho > 0$ and $\sigma > 0$, let $u \in S'$ with $f(u) \leqslant \inf f + \sigma\rho$. If $\mathbb{T}_\rho : S' \to S'$ is the map of Proposition 2.2 (cf. Remark 2.3), let $\tilde{u} = \mathbb{T}_\rho u \in S'$. Then $\|\tilde{u} - u^*\|_V < \rho$ and, taking into account (2.3), $f(\tilde{u}) \leqslant \inf f + \sigma\rho$. By applying Ekeland's variational principle on the complete metric space S' (see [**15**, Theorem 1.1]), we find $v \in S'$ such that $f(v) \leqslant f(\tilde{u}) \leqslant f(u)$, $\|v - \tilde{u}\| \leqslant \rho$ and $f(w) \geqslant f(v) - \sigma\|w - v\|$, for every $w \in S'$. As in inequality (2.5), it readily follows that $\|v - v^*\|_V \leqslant 2K\|v - \tilde{u}\| + \|\tilde{u} - u^*\|_V < (2K + 1)\rho$. Finally, $\|v - u\| \leqslant \|v - \tilde{u}\| + \|\mathbb{T}_\rho u - u\| \leqslant \rho + \|\mathbb{T}_\rho u - u\|$, concluding the proof. □

Note that, in Banach spaces, essentially conclusion (b) of Theorem 2.7 can be recovered by (e) of Theorem 2.5 with $p = 1$, since $\|v - \eta\| - \|w - \eta\| \geqslant -\|w - v\|$ for all $w \in X$.

On Banach spaces, we can state the following theorem.

THEOREM 2.8 (Symmetric Ekeland's principle, II). *Assume that X is a Banach space and that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semi-continuous functional bounded from below such that (2.3) holds. Moreover, assume that, for all $u \in \mathrm{dom}(f)$, there exists $\xi \in S$ such that $f(\xi) \leqslant f(u)$. Then, for every $\rho > 0$ and $\sigma > 0$, there exists $v \in X$ with*

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)\rho$;
(b) $f(w) \geqslant f(v) - \sigma\|w - v\|$ *for all* $w \in X$.

In addition, one can assume that $f(v) \leqslant f(u)$ and $\|v - u\| \leqslant \rho + \|\mathbb{T}_\rho u - u\|$, where $u \in S$ is some element that satisfies $f(u) \leqslant \inf f + \sigma\rho$.

Proof. Let $u \in \mathrm{dom}(f)$ with $f(u) \leqslant \inf f + \sigma\rho$. Then let $\xi \in S$ with $f(\xi) \leqslant \inf f + \sigma\rho$. At this stage one can proceed as in the proof of Theorem 2.7, with Ekeland's principle now applied to f defined on the whole X, yielding a $v \in X$ with the desired properties. □

Now let $f, f_h : X \to \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous functionals such that

$$\text{for any } u \in \mathrm{dom}(f) \cap S, \text{ there is } (u_h) \subset S \text{ with } u_h \longrightarrow u \text{ and } f_h(u_h) \longrightarrow f(u) \tag{2.8}$$

and

$$\liminf_h \Big(\inf_X f_h\Big) \geqslant \inf_X f. \tag{2.9}$$

As pointed out in [**9**], in some sense, this means that the function f is the uniform Γ-limit of the sequence (f_h). In the framework of Definition 2.1, we introduce the following definition.

Definition 2.9. We set $X_{\mathcal{H}_*} := \{u \in S : u^H = u, \text{ for all } H \in \mathcal{H}_*\}$.

Remark 2.10. In the framework of Definition 2.1, the space $(X_{\mathcal{H}_*}, \|\cdot\|)$ is complete, as it is closed in X. Conversely, assume only that the conclusion of the symmetric Ekeland principle holds true for the subclass of lower semi-continuous functionals $f : (X, \|\cdot\|_V) \to \mathbb{R} \cup \{+\infty\}$ bounded from below and which are not increasing under polarization of elements $u \in S$. Then $(X_{\mathcal{H}_*}, \|\cdot\|_V)$ is complete if u^H is contractive with respect to $\|\cdot\|_V$. In fact, let (u_h) be a Cauchy sequence in $(X_{\mathcal{H}_*}, \|\cdot\|_V)$. Defining $f : X \to \mathbb{R}^+$ by $f(u) := \lim_j \|u_j - u\|_V$, then f is continuous and $f(u_h) \to 0$ as $h \to \infty$, yielding $\inf f = 0$. Observe also that, by contractivity,

$$f(u^H) = \lim_j \|u_j - u^H\|_V = \lim_j \|u_j^H - u^H\|_V \leqslant \lim_j \|u_j - u\|_V = f(u),$$

for all $H \in \mathcal{H}_*$ and $u \in S$ and, for all $u \in X$,

$$f(\Theta(u)) = \lim_j \|u_j - \Theta(u)\|_V = \lim_j \|\Theta(u_j) - \Theta(u)\|_V \leqslant \lim_j \|u_j - u\|_V = f(u).$$

Given $\varepsilon \in (0,1)$, there is $v \in X$ with $f(v) \leqslant \varepsilon^2$, $\|v - v^*\|_V < \varepsilon$ and $f(w) \geqslant f(v) - \varepsilon\|w - v\|_V$, for all $w \in X$. By choosing $w = u_j$ and letting $j \to \infty$, it holds $f(v) \leqslant \varepsilon f(v)$, namely, $\|u_h - v\|_V \to 0$ as $h \to \infty$. Moreover, $v = v^*$, by the arbitrariness of ε. Hence, $v \in \mathcal{H}_*$.

Under the above conditions (2.8)–(2.9), we have a symmetric version of an Ekeland-type principle proposed by Corvellec [**9**, Proposition 1].

Theorem 2.11 (Symmetric Ekeland's principle, III). *Assume that X is a Banach space and that $f, f_h : X \to \mathbb{R} \cup \{+\infty\}$ are proper lower semi-continuous functionals with f, f_h bounded from below satisfying conditions* (2.8)–(2.9). *Moreover, assume that*

$$f_h(u^H) \leqslant f_h(u) \quad \text{for all } u \in S,\ H \in \mathcal{H}_* \text{ and } h \in \mathbb{N}. \tag{2.10}$$

Let Y be a nonempty subset of S, $\rho > 0$ and $\sigma > 0$ such that

$$\inf_Y f < \inf_X f + \sigma\rho.$$

Then, for every $h_0 \geqslant 1$, there exist $h \geqslant h_0$, $m > 1$, $(u_h) \subset S$ and $(v_h) \subset X$ such that

(a) $\|v_h - v_h^*\|_V < (K(C_\Theta + 1) + 1)\rho$;
(b) $|f_h(v_h) - \inf_X f| < \sigma\rho$;
(c) $d(v_h, Y) < \rho + \|T_{(m-1)\rho/m} u_h - u_h\|$;
(d) $f_h(w) \geqslant f_h(v_h) - \sigma\|w - v_h\|$ *for all* $w \in X$.

In particular, if $f_h = f$ for all $h \in \mathbb{N}$ and $Y \subset X_{\mathcal{H}_*}$, then there exists $v \in X$ such that

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)\rho$;
(b) $|f(v) - \inf_X f| < \sigma\rho$;

(c) $d(v, Y) < \rho$;
(d) $f(w) \geqslant f(v) - \sigma\|w - v\|$ *for all* $w \in X$.

Proof. Given $h_0 \geqslant 1$, $\rho > 0$ and $\sigma > 0$, taking into account (2.8)–(2.9), arguing as in the proof of Corvellec [**9**, Proposition 1], one finds $u \in Y \cap \mathrm{dom}(f)$, $m > 1$, $\hat\sigma \in (0, \sigma)$, $\tilde\sigma \in (\hat\sigma, \sigma)$ with $m\tilde\sigma/(m-1) < \sigma$ and $f(u) < \inf f + \hat\sigma\rho$, and points $u_h \in S \cap \mathrm{dom}(f_h)$ such that

$$\|u_h - u\| < \rho/m, \quad \inf_X f_h \geqslant \inf_X f - \frac{(\tilde\sigma - \hat\sigma)\rho}{2}, \quad f_h(u_h) \leqslant f(u) + \frac{(\tilde\sigma - \hat\sigma)\rho}{2},$$

and, in turn,

$$f_h(u_h) < \inf_X f_h + \tilde\sigma\rho. \tag{2.11}$$

By means of (2.10) condition (2.3) is satisfied for the functionals f_h. Therefore, in light of Theorem 2.8 (applied to f_h, starting from the point u_h: see (2.11)), with σ replaced by $m\tilde\sigma/(m-1)$ and ρ replaced by $(m-1)\rho/m$, respectively, there exist $v_h \in X$ such that

$$f_h(v_h) \leqslant f_h(u_h), \quad \|v_h - v_h^*\|_V < (K(C_\Theta + 1) + 1)\frac{m-1}{m}\rho < (K(C_\Theta + 1) + 1)\rho,$$
$$f_h(w) \geqslant f_h(v_h) - \frac{m}{m-1}\tilde\sigma\|w - v_h\| \geqslant f_h(v_h) - \sigma\|w - v_h\|, \quad \text{for all } w \in X.$$

Also, it holds $|f_h(v_h) - \inf f| < \sigma\rho$, since

$$\inf_X f - \sigma\rho < \inf_X f - \frac{(\tilde\sigma - \hat\sigma)\rho}{2} \leqslant f_h(v_h) \leqslant f_h(u_h) \leqslant f(u) + \frac{(\tilde\sigma - \hat\sigma)\rho}{2} < \inf f + \sigma\rho.$$

Moreover, noting that $\|v_h - u_h\| < (m-1)\rho/m + \|T_{(m-1)\rho/m}u_h - u_h\|$, it holds

$$d(v_h, Y) \leqslant \|v_h - u\| \leqslant \|v_h - u_h\| + \|u_h - u\| < \rho + \|T_{(m-1)\rho/m}u_h - u_h\|.$$

The last conclusion of the statement can be easily obtained by taking into account that $T_\rho u = u$ for all $\rho > 0$ and $u \in Y \subset X_{\mathcal{H}_*}$. □

Based upon the strong Ekeland's principle stated by Georgiev [**20**], which exhibits some additional *stability features*, we formulate the following symmetric version.

Theorem 2.12 (Symmetric Ekeland's principle, IV). *Assume that* X *is a Banach space and that* $f : X \to \mathbb{R} \cup \{+\infty\}$ *is a proper and lower semi-continuous functional bounded from below such that* (2.3) *holds. Then, for every* $\rho_1, \rho_2 > 0$, $\sigma > 0$ *and* $u \in S$, *such that*

$$f(u) < \inf_X f + \sigma\rho_1,$$

there exists a point $v \in X$ *such that*

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)(\rho_1 + \rho_2)$;
(b) $f(w) \geqslant f(v) - \sigma\|w - v\|$ *for all* $w \in X$;
(c) *for every sequence* $(u_h) \subset X$ *it follows*

$$\lim_h (f(u_h) + \sigma\|u_h - v\|) = f(v) \quad \Rightarrow \quad \lim_h u_h = v.$$

Proof. Given $\rho_1, \rho_2 > 0$ and $\sigma > 0$, let $u \in S$ be such that $f(u) < \inf f + \sigma\rho_1$. If $\mathbb{T}_\rho : S \to S$ is the map of Proposition 2.2, let $\tilde u = \mathbb{T}_{\rho_1+\rho_2}u \in S$. Then $\|\tilde u - u^*\|_V < \rho_1 + \rho_2$ and, taking into account (2.3), $f(\tilde u) < \inf f + \sigma\rho_1$. By Georgiev [**20**, Theorem 1.6] there exists $v \in X$ such that (b) and (c) hold and $\|v - \tilde u\| < \rho_1 + \rho_2$. Then $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)(\rho_1 + \rho_2)$, by arguing as in the previous proofs. □

In some situations, a version of Ekeland's variational principle, sometimes called *altered* principle, has been found very useful [**26**]. Here follows a symmetric version of it. A similar statement holds with S in place of X, when S is closed.

Theorem 2.13 (Symmetric Ekeland's principle, V). *Assume that X is a Banach space and that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper and lower semi-continuous bounded-below functional such that* (2.3) *holds. Then, for every $u \in S$, $\rho > 0$ and $\sigma > 0$, there exists an element $v \in X$ such that*

(a) $f(w) > f(v) - \sigma\|w - v\|$ *for all* $w \in X \setminus \{v\}$;
(b) $f(v) \leqslant f(u) - \sigma\|v - \mathbb{T}_\rho u\|$.

If in addition $u \in X_{\mathcal{H}_*}$, then (b) *strengthens to* $f(v) \leqslant f(u) - \sigma\|v - u\|$.

Proof. Given $u \in S$, $\rho > 0$ and $\sigma > 0$, consider $\mathbb{T}_\rho u \in S$. By applying [**26**, Theorem A, p. 814] to $\mathbb{T}_\rho u$ and taking into account that $f(\mathbb{T}_\rho u) \leqslant f(u)$ by (2.3), we get an element $v \in X$ satisfying properties (a) and (b). □

Remark 2.14. Let $u \in S$ be such that $f(u) \leqslant \inf f + \rho\sigma$, for some $\rho, \sigma > 0$. Then, in addition to the conclusions of Theorem 2.13, it follows $\|v - v^*\|_V \leqslant \rho$, as in the previous statements. In fact, in light of (b) of Theorem 2.13, we have

$$\|v - \mathbb{T}_\rho u\| \leqslant \frac{f(u) - f(v)}{\sigma} \leqslant \frac{f(u) - \inf f}{\sigma} \leqslant \rho,$$

which in turn allows us to get the desired conclusion, taking into account that $\|\mathbb{T}_\rho u - u^*\|_V < \rho$. Also one has $\|v - u\| \leqslant \rho + \|\mathbb{T}_\rho u - u\|$. In other words, Theorem 2.13 is stronger than the previous statements owing to the fact that it holds for *any* point $u \in S$. On the other hand, the price to be paid is that the *location* of v with respect to u is *no longer available* and it is recovered, provided that $f(u) \leqslant \inf f + \rho\sigma$.

Let X' denote the topological dual space of X. We need to recall from [**13**] the following definition.

Definition 2.15. Let X be a Banach space, β be a family of bounded subsets of X which constitutes a bornology, $f : X \to \mathbb{R} \cup \{+\infty\}$ be a functional and $u \in \mathrm{dom}(f)$. We say that f is β-differentiable at u with β-derivative $\varphi = f'(u) \in X'$ if

$$\lim_{t \to 0} \frac{f(u + tw) - f(u) - \langle \varphi, tw \rangle}{t} = 0$$

uniformly for w inside the elements of β. We denote by τ_β the topology on X' of uniform convergence on the elements of β.

When β is the class of all bounded subsets of X, then the β-differentiability coincides with Fréchet differentiability and τ_β coincides with the norm topology on X'. When β is the class of all singletons of X, the β-differentiability coincides with Gateaux differentiability and τ_β is the weak* topology on X'.

We consider the Banach space $(X_\beta, \|\cdot\|_\beta)$ defined as follows:

$$X_\beta := \{g : X \longrightarrow \mathbb{R} : \ g \text{ is bounded, Lipschitzian and } \beta\text{-differentiable on } X\},$$
$$\|g\|_\beta := \|g\|_\infty + \|g'\|_\infty, \quad \|g\|_\infty = \sup_{u\in X} |g(u)|, \quad \|g'\|_\infty = \sup_{u\in X} \|g'(u)\|.$$

DEFINITION 2.16. We say that $b \in X_\beta$ is a bump function if $\mathrm{supt}(b) \neq \emptyset$ is bounded.

Next, we recall a localized version of Deville–Godefroy–Zizler's variational principle (see [**13**, Corollary II.4 and Remark II.5]).

THEOREM 2.17 (Deville–Godefroy–Zizler's principle). *Assume that X is a Banach space that admits a bump function in X_β and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous functional bounded from below. Then there exists a positive number $\mathcal{A}$ such that, for all $\varepsilon \in (0, 1)$, and $u \in X$ with $f(u) < \inf f + \mathcal{A}\varepsilon^2$, there exist $g \in X_\beta$ and $v \in X$ such that*

(a) $\|v - u\| \leqslant \varepsilon$;
(b) $\|g\|_\infty \leqslant \varepsilon$ *and* $\|g'\|_\infty \leqslant \varepsilon$;
(c) $f(w) + g(w) \geqslant f(v) + g(v)$ *for all* $w \in X$.

Next, we state a symmetric version of Deville–Godefroy–Zizler's variational principle.

THEOREM 2.18 (Symmetric Deville–Godefroy–Zizler's principle). *Assume that X is a Banach space that admits a bump function in X_β and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous functional bounded from below satisfying* (2.3). *Then there exists a positive number $\mathcal{A}$ such that, for all $\varepsilon \in (0, 1)$, and $u \in S$ with $f(u) < \inf f + \mathcal{A}\varepsilon^2$, there exist $g \in X_\beta$ and $v \in X$ such that*

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)\varepsilon$;
(b) $\|v - u\| \leqslant \varepsilon + \|\mathbb{T}_\varepsilon u - u\|$;
(c) $\|g\|_\infty \leqslant \varepsilon$ *and* $\|g'\|_\infty \leqslant \varepsilon$;
(d) $f(w) + g(w) \geqslant f(v) + g(v)$ *for all* $w \in X$.

Proof. By Theorem 2.17 there exists a positive number $\mathcal{A}$ with the stated properties. Let $u \in S$ and $\varepsilon \in (0, 1)$ such that $f(u) < \inf f + \mathcal{A}\varepsilon^2$. If $\mathbb{T}_\varepsilon : S \to S$ is as in Proposition 2.2, then we set $\tilde{u} := \mathbb{T}_\varepsilon u \in S$. By construction we have $\|\tilde{u} - u^*\|_V < \varepsilon$ and $f(\tilde{u}) < \inf f + \mathcal{A}\varepsilon^2$. Hence, by the just stated principle, there are $g \in X_\beta$, with $\|g\|_\infty \leqslant \varepsilon$, and $\|g'\|_\infty \leqslant \varepsilon$ and $v \in X$ such that $\|v - \tilde{u}\| \leqslant \varepsilon$ and $f(w) + g(w) \geqslant f(v) + g(v)$ for every $w \in X$. Furthermore, we have $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)\varepsilon$ with the usual argument, as well as $\|v - u\| \leqslant \|v - \tilde{u}\| + \|\tilde{u} - u\| \leqslant \varepsilon + \|\mathbb{T}_\varepsilon u - u\|$. This concludes the proof. □

2.3. *Statements with weights*

In this section, we derive a symmetric version of Ekeland's variational principle with weights (see also [**17**, **29**, **32**]), based upon the following result due to Zhong (take $x_0 = y$ in [**32**, Theorem 1.1]). The result is often used to prove that a lower semi-continuous bounded-below functional that satisfies a suitable weighted PS condition needs to be coercive.

THEOREM 2.19 (Zhong's principle). *Let X be a complete metric space and consider a nondecreasing and continuous function $h : [0, +\infty) \to [0, +\infty)$ such that*

$$\int_0^{+\infty} \frac{1}{1 + h(s)}\, ds = +\infty.$$

Assume that $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semi-continuous functional bounded from below. Let $u \in X$, $\rho > 0$ and $\sigma > 0$ such that

$$f(u) < \inf_X f + \sigma\rho.$$

Then there exists $v \in X$ such that

(a) $f(v) \leqslant f(u)$;
(b) $d(v, u) \leqslant r(\rho)$;
(c) $f(w) \geqslant f(v) - \sigma(d(w, v))/(1 + h(d(v, u)))$ *for all* $w \in X$,

where $r(\rho)$ is a positive number that satisfies

$$\int_0^{r(\rho)} \frac{1}{1 + h(s)}\, ds \geqslant \rho.$$

As a consequence, in the framework of Definition 2.1, we obtain the following theorem.

THEOREM 2.20 (Symmetric Zhong's principle). *Let X be a Banach space and consider a nondecreasing continuous function $h : [0, +\infty) \to [0, +\infty)$ such that*

$$\int_0^{+\infty} \frac{1}{1 + h(s)}\, ds = +\infty.$$

Assume that, for $\rho_0 > 0$ sufficiently small, there exists a function $r : [0, \rho_0) \to [0, \infty)$ with

$$\int_0^{r(\rho)} \frac{1}{1 + h(s)}\, ds \geqslant \rho, \qquad \lim_{\rho \to 0^+} r(\rho) = 0. \tag{2.12}$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous functional bounded from below such that condition (2.3) holds. Let $u \in S$, $\rho > 0$ and $\sigma > 0$ be such that

$$f(u) < \inf_X f + \sigma\rho. \tag{2.13}$$

Then there exists $v \in X$ such that

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)r(\rho)$;
(b) $f(v) \leqslant f(u)$;
(c) $\|v - u\| \leqslant r(\rho) + \|\mathbb{T}_{r(\rho)}u - u\|$;
(d) $f(w) \geqslant f(v) - \sigma(\|w - v\|)/(1 + h(\|v - \mathbb{T}_{r(\rho)}u\|))$ *for every* $w \in X$.

Proof. Let $u \in S$, $\rho > 0$ and $\sigma > 0$ with $f(u) < \inf f + \sigma\rho$. Let also $r(\rho)$ be a positive number that satisfies conditions (2.12). Then, if $\mathbb{T}_{r(\rho)} : S \to S$ is the map of Proposition 2.2, let $\tilde{u} := \mathbb{T}_{r(\rho)}u \in S$. Then $\|\tilde{u} - u^*\|_V < r(\rho)$ and, taking into account (2.3), we can conclude $f(\tilde{u}) < \inf f + \sigma\rho$. By applying Theorem 2.19 to this element $\tilde{u}$, we find an element $v \in X$ such that $\|v - \tilde{u}\| \leqslant r(\rho)$, $f(v) \leqslant f(\tilde{u}) \leqslant f(u)$ and

$$f(w) \geqslant f(v) - \sigma \frac{\|w - v\|}{1 + h(\|v - \mathbb{T}_{r(\rho)}u\|)} \quad \text{for every } w \in X.$$

Also, we have

$$\|v - u\| \leqslant \|v - \tilde{u}\| + \|\mathbb{T}_{r(\rho)} - u\| \leqslant r(\rho) + \|\mathbb{T}_{r(\rho)} - u\|.$$

We conclude with $\|v - v^*\|_V \leqslant K(C_\Theta + 1)\|v - \tilde{u}\| + \|\tilde{u} - u^*\|_V < (K(C_\Theta + 1) + 1)r(\rho)$. □

REMARK 2.21. In the case $h \equiv 0$, one finds precisely the symmetric version of the classical Ekeland's variational principle (note that one can take $r(\rho) = \rho$). In the Cerami case $h(s) = s$ (see [7]), one can take $r(\rho) = e^\rho - 1$ and the conclusion of Theorem 2.20 reads as follows: for every $u \in S$ that satisfies (2.13), with $\rho > 0$ and $\sigma > 0$, there exists $v \in X$ such that

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)(e^\rho - 1)$;
(b) $f(v) \leqslant f(u)$;
(c) $\|v - u\| \leqslant e^\rho - 1 + \|\mathbb{T}_{e^\rho - 1}u - u\|$;
(d) $f(w) \geqslant f(v) - \sigma(\|w - v\|)/(1 + \|v - \mathbb{T}_{e^\rho - 1}u\|)$ for all $w \in X$.

Furthermore, if $u \in X_{\mathcal{H}_*}$ and $\rho = \sigma > 0$, then there exists $v \in X$ such that

(a) $\|v - v^*\|_V < (K(C_\Theta + 1) + 1)(e^\rho - 1)$;
(b) $f(v) \leqslant f(u)$;
(c) $\|v - u\| \leqslant e^\rho - 1$;
(d) $f(w) \geqslant f(v) - \rho(\|w - v\|)/(1 + \|v - u\|)$ for all $w \in X$.

Next, we highlight some by-products of the previous principles in the context of nonsmooth critical point theory. We recall the definition of weak slope [12]. The symbol $B(u, \delta)$ stands for the open ball in X with centre u and radius δ, and $\mathrm{epi}(f) = \{(u, \lambda) \in X \times \mathbb{R} : f(u) \leqslant \lambda\}$.

DEFINITION 2.22. For every $u \in X$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the values of σ in $[0, \infty)$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H} : \ B((u, f(u)), \delta) \cap \mathrm{epi}(f) \times [0, \delta] \longrightarrow X,$$

satisfying, for all $(\xi, \mu) \in B((u, f(u)), \delta) \cap \mathrm{epi}(f)$ and $t \in [0, \delta]$,

$$\|\mathcal{H}((\xi, \mu), t) - \xi\| \leqslant t, \quad f(\mathcal{H}((\xi, \mu), t)) \leqslant f(\xi) - \sigma t.$$

The extended real number $|df|(u)$ is called the weak slope of f at u.

REMARK 2.23. If f is of class C^1, then $|df|(u) = \|df(u)\|$; see [12, Corollary 2.12]. If $u \in X$, with $f(u) < +\infty$, the strong slope of f at u (see [11]) is the extended real $|\nabla f|(u)$,

$$|\nabla f|(u) := \begin{cases} \limsup\limits_{\xi \to u} \dfrac{f(u) - f(\xi)}{d(u, \xi)} & \text{if } u \text{ is not a local minimum for } f; \\ 0 & \text{if } u \text{ is a local minimum for } f. \end{cases}$$

It easily follows from the definition that $|df|(u) \leqslant |\nabla f|(u)$.

We can now state the following corollary.

COROLLARY 2.24. *Let X be a Banach space and $h : [0, +\infty) \to [0, +\infty)$ be a nondecreasing and continuous function such that*

$$\int_0^{+\infty} \frac{1}{1 + h(s)}\, ds = +\infty.$$

Assume that, for $\rho_0 > 0$ sufficiently small, there exists a function $r : [0, \rho_0) \to [0, \infty)$ with

$$\int_0^{r(\rho)} \frac{1}{1 + h(s)}\, ds \geqslant \rho, \qquad \lim_{\rho\to 0^+} r(\rho) = 0.$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous functional bounded from below such that (2.3) holds. Then, for every $\rho > 0$ and $u_\rho \in S$ with

$$f(u_\rho) < \inf_X f + \rho^2,$$

there exists $v_\rho \in X$ such that

(a) $\|v_\rho - v_\rho^*\|_V < (K(C_\Theta + 1) + 1)r(\rho)$;
(b) $(1 + h(\|v_\rho - \mathbb{T}_{r(\rho)} u_\rho\|))|df|(v_\rho) \leqslant \rho$ *for all $w \in X$.*

In particular, for every minimizing sequence $(u_j) \subset S$ for f, there exists a minimizing sequence $(v_j) \subset X$ and $(\mu_j) \subset \mathbb{R}^+$ with $\mu_j \to 0$ such that

$$\lim_{j\to\infty} \|v_j - v_j^*\|_V = 0, \qquad \lim_{j\to\infty} (1 + h(\|v_j - \mathbb{T}_{\mu_j} u_j\|))|df|(v_j) = 0.$$

Moreover, for every symmetric minimizing sequence $(u_j) \subset X_{\mathcal{H}_*}$ for f, there exists a minimizing sequence $(v_j) \subset X$ such that

$$\lim_{j\to\infty} \|v_j - v_j^*\|_V = 0, \qquad \lim_{j\to\infty} (1 + h(\|v_j - u_j\|))|df|(v_j) = 0.$$

Proof. Taking into account Remark 2.23, it is an easy consequence of Theorem 2.20. □

2.4. *Statements with constraints*

A symmetric version of Ekeland's principle *with constraints*, in the spirit of Ekeland [**15**, Theorem 3.1], can also be formulated. Assume that $G_j : X \to \mathbb{R}$ with $1 \leqslant j \leqslant m$ are C^1 functions, let $1 \leqslant p \leqslant m$ and consider the set

$$\mathscr{C} = \{u \in X : \ G_j(u) = 0 \quad \text{for } 1 \leqslant j \leqslant p \quad \text{and } G_j(u) \geqslant 0 \quad \text{for } p+1 \leqslant j \leqslant m\}.$$

For all u in $\mathscr{C}$, we denote by $\mathscr{I}(u)$ the index set of saturated constraints (cf. [**15**]), namely, $j \in \mathscr{I}(u)$ if and only if $G_j(u) = 0$. We consider the following assumptions:

$$f : X \to \mathbb{R} \text{ is Fréchet differentiable}, \quad -\infty < \inf_{\mathscr{C}} f < +\infty; \tag{2.14}$$

$$\text{for all } u \in \mathscr{C} \text{ there exists } \xi \in \mathscr{C} \cap S \text{ such that } f(\xi) \leqslant f(u); \tag{2.15}$$

$$\text{for all } u \in \mathscr{C} \text{ the elements } \{dG_j(u)\}_{j\in\mathscr{I}(u)} \text{ are linearly independent in} X'; \tag{2.16}$$

$$\begin{cases} \forall u \in \mathscr{C} \cap S, \quad \forall H \in \mathcal{H}_* : \ u^H \in \mathscr{C}, \\ \forall u \in \mathscr{C} \cap S, \quad \forall H \in \mathcal{H}_* : \ f(u^H) \leqslant f(u). \end{cases} \tag{2.17}$$

Then, for every $\varepsilon > 0$, there exists $u_\varepsilon \in \mathscr{C}$ such that

$$f(u_\varepsilon) \leqslant \inf_{\mathscr{C}} f + \varepsilon^2, \quad \|u_\varepsilon - u_\varepsilon^*\|_V < C\varepsilon, \quad \left\| df(u_\varepsilon) - \sum_{j=1}^m \lambda_j dG_j(u_\varepsilon) \right\|_{X'} \leqslant \varepsilon,$$

for some $\lambda_j \in \mathbb{R}$, $1 \leqslant j \leqslant m$, such that $\lambda_j \geqslant 0$ for $p+1 \leqslant j \leqslant m$ and $\lambda_j = 0$ if $G_j(u_\varepsilon) \neq 0$. The assertion follows by applying Theorem 2.8 to the functional $\hat{f} : X \to \mathbb{R} \cup \{+\infty\}$,

$$\hat{f}(u) := \begin{cases} f(u) & \text{for } u \in \mathscr{C}, \\ +\infty & \text{for } u \in X \setminus \mathscr{C}, \end{cases}$$

finding almost symmetric point $u_\varepsilon \in \mathscr{C}$ such that $f(u_\varepsilon) \leqslant \inf f|_{\mathscr{C}} + \varepsilon^2$ and

$$\forall w \in \mathscr{C}: \quad f(w) \geqslant f(u_\varepsilon) - \varepsilon\|w - u_\varepsilon\|,$$

and then arguing exactly as in the proof of Ekeland [**15**, Theorem 3.1], namely using [**15**, Lemmas 3.2 and 3.3], in view of assumptions (2.14)–(2.16). The assumptions of Theorem 2.8 are fulfilled since $\hat{f}$ is lower semi-continuous, bounded from below (f being bounded from below on $\mathscr{C}$) and, in light of (2.17), it satisfies $\hat{f}(u^H) \leqslant \hat{f}(u)$ for every $u \in S$ and $H \in \mathcal{H}_*$. Also, by virtue of (2.15), for all $u \in \mathrm{dom}(\hat{f})$ there exists $\xi \in S$ such that $\hat{f}(\xi) \leqslant \hat{f}(u)$. In the case of a single constraint, namely $m = 1$, then assumption (2.16) reads: $G(u) = 0$ implies $dG(u) \neq 0$. On the concrete side, (2.17) is satisfied in various situations, meaningful in the calculus of variations, such as $G : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$, $G(u) = \int_{\mathbb{R}^N} H(|u|) - 1$ for suitable $H \in C^1(\mathbb{R})$ and functionals $f : W^{1,p}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ discussed in Section 3.1.1.

2.5. *Symmetry, coercivity and PS conditions*

A sequence $(u_h) \subset X$ is said to be a PS sequence for $f \in C^1(X)$ if $(f(u_h))$ is bounded and $\|df(u_h)\|_{X'} \to 0$, as $h \to \infty$. Also we say that f satisfies the PS condition, if each PS sequence admits a converging subsequence. If f is bounded from below and satisfies the PS condition, then it is coercive [**5**, **9**], meaning that

$$\liminf_{\|u\| \to +\infty} f(u) = +\infty.$$

Actually, an even more general property holds and it is sufficient to assume that the PSB condition holds, namely, each $(u_h) \subset X$ with $(f(u_h))$ bounded and $\|df(u_h)\|_{X'} \to 0$ is bounded (see [**9**, Corollary 1], for details). As pointed out in [**24**, Section 10], a typical argument to prove the above conclusion is based upon a clever application of Ekeland's principle, after observing that a violation of the coercivity yields $\ell \in \mathbb{R}$, $\ell \geqslant \inf f$ and a sequence $(u_h) \subset X$ such that $f(u_h) \leqslant \ell + \gamma_h$ and $\|u_h\| \geqslant h$, where $(\gamma_h) \subset \mathbb{R}^+$ is a given sequence with $\gamma_h \to 0$ as $h \to \infty$. Note that $\ell + \gamma_h - \inf f > 0$ for all $h \in \mathbb{N}$. Let $\sigma_h > 0$ with $\sigma_h \to 0$ as $h \to \infty$ and $\rho_h = (\ell + \gamma_h - \inf f)/\sigma_h > 0$ be such that $\rho_h \leqslant h/2$, yielding

$$f(u_h) \leqslant \inf f + \sigma_h \rho_h, \quad h \in \mathbb{N}.$$

Under reasonable assumptions, we can also have $(u_h) \subset S$. At this stage, if (2.3) holds,

$$f(\mathbb{T}_{\rho_h} u_h) \leqslant f(u_h) \leqslant \inf f + \sigma_h \rho_h, \quad \|\mathbb{T}_{\rho_h} u_h - u_h^*\|_V < \rho_h.$$

Then Ekeland's principle yields $(v_h) \subset X$ with $f(v_h) \leqslant f(u_h) \leqslant \ell + \gamma_h$, $\|df(v_h)\|_{X'} \leqslant \sigma_h$ and $\|v_h - \mathbb{T}_{\rho_h} u_h\| \leqslant \rho_h$, implying $\|v_h - v_h^*\|_V < C\rho_h$. Note that, assuming $\|u^H\| = \|u\|$ for all $u \in S$ and $H \in \mathcal{H}_*$, which is reasonable for applications to PDEs, there holds $\|v_h\| \geqslant \|\mathbb{T}_{\rho_h} u_h\| - \rho_h = \|u_h\| - \rho_h \geqslant h/2$, yielding $\|v_h\| \to \infty$, as $h \to \infty$. In particular, it follows that $f(v_h) \to \ell$, since

$$\ell = \liminf_{\|u\| \to +\infty} f(u) \leqslant \liminf_h f(v_h) \leqslant \limsup_h f(v_h) \leqslant \lim_h (\ell + \gamma_h) = \ell.$$

Since $\sigma_h \to 0$, (v_h) in an unbounded PS sequence, contradicting the PSB condition. To guarantee that, in addition, $\|v_h - v_h^*\|_V \to 0$, one would need that $\rho_h \to 0$. On the other hand, $\rho_h, \sigma_h, \gamma_h \to 0$, by $\sigma_h \rho_h = \ell + \gamma_h - \inf f$, yields $\ell = \inf f$, which is not the case in general. In conclusion, this argument does not seem to allow obtaining a true unbounded *almost symmetric PS sequence*, which would of course considerably improve the statement on coercivity, replacing PSB with some symmetric version of it involving PS sequences (u_h) with $\|u_h - u_h^*\|_V \to 0$. If f is bounded from below, (2.3) holds and it satisfies the symmetric PSB condition, then

$$\liminf_{\|u\| \to +\infty} f(u) > \inf_X f.$$

It is sufficient to argue by contradiction and let $\ell = \inf f$ in the previous proof, allowing $\rho_h, \sigma_h, \gamma_h \to 0$. The relationships between the symmetry of the functional, its coercivity and PS conditions of some kind would deserve further attention.

2.6. *Symmetric quasi-convex PS sequences*

To the author's knowledge, the next notion was first introduced by Bartsch and Degiovanni [**2**].

Definition 2.25. Let X be a Banach space, $f : X \to \mathbb{R}$ be a lower semi-continuous functional and $u \in X$. We define the functional $Q_u : X \to \bar{\mathbb{R}}$ by setting

$$Q_u(w) := \limsup_{\substack{z \to u \\ \zeta \to w \\ t \to 0}} \frac{f(z + t\zeta) + f(z - t\zeta) - 2f(z)}{t^2}, \quad \text{for every } w \in X.$$

In the framework of Definition 2.1, we also introduce the following definition.

Definition 2.26. Let X be a Banach space and $f : X \to \mathbb{R}$ be a lower semi-continuous functional. We say that $(u_h) \subset X$ is a *symmetric quasi-convex PS sequence* at level $c \in \mathbb{R}$ ($(\mathrm{SQPS})_c$ -sequence, in short) if

$$\lim_{h\to\infty} f(u_h) = c, \quad \lim_{h\to\infty} |df|(u_h) = 0,$$

and, in addition,

$$\lim_{h\to\infty} \|u_h - u_h^*\|_V = 0, \quad \liminf_{h\to\infty} Q_{u_h}(w) \geqslant 0, \quad \forall w \in X. \tag{2.18}$$

We say that f satisfies the *symmetric quasi-convex PS condition* at level c, $(SQPS)_c$, in short, if every $(SQPS)_c$-sequence that admits a subsequence strongly converging in W, up to a subsequence, converges strongly in X.

Compared to a standard PS sequence, two additional items of information are involved on (u_h), a quasi-symmetry and a quasi-convexity condition.

Remark 2.27. As pointed out in [**18**, **23**], the fact that a PS sequence $(u_h) \subset X$ for a functional $f : X \to \mathbb{R}$ of class C^2 satisfies the additional second-order condition

$$\liminf_{h\to\infty} \langle f''(u_h)w, w\rangle \geqslant 0, \quad \text{for all } w \in X,$$

can sometimes be crucial for the proof of the strong convergence of (u_h) itself to some limit point $u \in X$. Furthermore, the additional symmetry condition $\|u_h - u_h^*\|_V \to 0$, as $h \to \infty$, usually provides compactifying effects (see, for example, [**30**, Section 4.2]). Based upon these considerations, it is quite clear that, in some sense, the $(SQPS)_c$-condition is much weaker than the standard PS condition. Of course, $Q_u(w) = \langle f''(u)w, w\rangle$ when f is of class C^2 and replacing $\langle f''(u)w, w\rangle$ with $Q_u(w)$ appears to be a natural extension when the function is not C^2 smooth.

Now let X be a Hilbert space and consider the following assumptions:

$$f(u^H) \leqslant f(u) \text{ for all } u \in S \text{ and } H \in \mathcal{H}_*; \tag{2.19}$$

$$\text{for all } X \text{ there exists } \xi \in S \text{ such that } f(\xi) \leqslant f(u); \tag{2.20}$$

$$\text{if } (u_h) \subset X \text{ is bounded, then } (\xi_h) \subset S \text{ is bounded}; \tag{2.21}$$

$$\|u^H\| \leqslant \|u\| \text{ for all } u \in S \text{ and } H \in \mathcal{H}_*; \tag{2.22}$$

$$f \text{ admits a bounded minimizing sequence.} \tag{2.23}$$

Note that assumptions (2.20)–(2.22) are satisfied in many typical concrete situations, like when X is a Sobolev space $W_0^{1,p}(\Omega)$, Ω is a ball or $\mathbb{R}^N$, S is the cone of its positive functions and the functional satisfies $f(|u|) \leqslant f(u)$ for all $u \in X$. Assumption (2.23) is mild but not automatically satisfied of course; for instance, all the minimizing sequences for the exponential function on $\mathbb{R}$ are unbounded.

We can now state the following theorem.

THEOREM 2.28. *Assume that $f : X \to \mathbb{R}$ is a lower semi-continuous functional bounded from below such that conditions (2.19)–(2.23) hold. Then f admits an $(SQPS)_{\inf f}$-sequence.*

Proof. In the course of the proof, C will denote a generic constant that might change from line to line. By means of assumption (2.23), we can find a bounded minimizing sequence $(u_h) \subset X$ for f, namely, there exists a sequence $(\varepsilon_h) \subset \mathbb{R}^+$, with $\varepsilon_h \to 0$ as $h \to \infty$, such that $\|u_h\| \leqslant C$ and $f(u_h) < \inf f + \varepsilon_h^3$ for all $h \in \mathbb{N}$. In light of assumptions (2.20)–(2.21), there exists a sequence $(\xi_h) \subset S$ such that $\|\xi_h\| \leqslant C$ and $f(\xi_h) < \inf f + \varepsilon_h^3$, for all $h \in \mathbb{N}$. Taking into account that any norm $\|\cdot\|$ on X is a Kadec norm and that assumption (2.19) holds, by Theorem 2.5 (symmetric Borwein–Preiss's principle) with $p = 2$, $\sigma_h = \rho_h = \varepsilon_h$, we find two sequences $(v_h) \subset X$ and $(\eta_h) \subset X$ such that $\|v_h - v_h^*\|_V < C\varepsilon_h$, $f(v_h) < \inf f + \varepsilon_h^3$ as well as

$$\|v_h - \xi_h\| < \varepsilon_h + \|\mathbb{T}_{\varepsilon_h}\xi_h - \xi_h\|, \quad \|\eta_h - \xi_h\| \leqslant \varepsilon_h + \|\mathbb{T}_{\varepsilon_h}\xi_h - \xi_h\|, \tag{2.24}$$

$$f(w) \geqslant f(v_h) + \varepsilon_h(\|v_h - \eta_h\|^2 - \|w - \eta_h\|^2), \quad \text{for all } w \in X. \tag{2.25}$$

Fixed any $\zeta \in X$ and $t \in \mathbb{R}$, substituting $w := v_h + t\zeta$ and $w := v_h - t\zeta$ into (2.25) yields

$$f(v_h + t\zeta) \geqslant f(v_h) + \varepsilon_h(\|v_h - \eta_h\|^2 - \|v_h - \eta_h + t\zeta\|^2),$$
$$f(v_h - t\zeta) \geqslant f(v_h) + \varepsilon_h(\|v_h - \eta_h\|^2 - \|v_h - \eta_h - t\zeta\|^2).$$

Whence, taking into account the parallelogram law, it holds

$$f(v_h + t\zeta) + f(v_h - t\zeta) - 2f(v_h) \geqslant -2\varepsilon_h t^2 \|\zeta\|^2, \quad \text{for all } \zeta \in X \text{ and } t \in \mathbb{R}. \tag{2.26}$$

In turn, for every $w \in X$, it holds

$$\begin{aligned} Q_{v_h}(w) &= \limsup_{\substack{z \to v_h \\ \zeta \to w \\ t \to 0}} \frac{f(z + t\zeta) + f(z - t\zeta) - 2f(z)}{t^2} \\ &\geqslant \limsup_{\substack{\zeta \to w \\ t \to 0}} \frac{f(v_h + t\zeta) + f(v_h - t\zeta) - 2f(v_h)}{t^2} \\ &\geqslant -2\varepsilon_h \|w\|^2, \end{aligned}$$

which yields the desired property on Q_{v_h}. Note also that, from (2.25), for every h and $w \neq v_h$,

$$\frac{f(v_h) - f(w)}{\|w - v_h\|} \leqslant \varepsilon_h \frac{\|w - \eta_h\|^2 - \|v_h - \eta_h\|^2}{\|w - v_h\|} \leqslant \varepsilon_h(\|w - \eta_h\| + \|v_h - \eta_h\|).$$

By repeatedly applying (2.22), we get $\|\mathbb{T}_{\varepsilon_h}\xi_h\| \leqslant \|\xi_h\|$. Whence, recalling (2.24), it follows that

$$\begin{aligned}|df|(v_h) \leqslant |\nabla f|(v_h) &= \limsup_{w\to v_h} \frac{f(v_h)-f(w)}{\|w-v_h\|} \leqslant 2\varepsilon_h\|v_h-\eta_h\| \\ &\leqslant 2\varepsilon_h\|v_h-\xi_h\| + 2\varepsilon_h\|\xi_h-\eta_h\| \\ &< 4\varepsilon_h(\varepsilon_h + \|\mathbb{T}_{\varepsilon_h}\xi_h-\xi_h\|) \leqslant 4\varepsilon_h^2 + 8\varepsilon_h\|\xi_h\| \leqslant C\varepsilon_h.\end{aligned}$$

This concludes the proof. □

In the framework of Definition 2.1, we also introduce the following definition.

Definition 2.29. We set $X_* := \{u \in S : u^* = u\}$ and we say that X is symmetrically embedded into W if $\|u^*\| \leqslant \|u\|$ for all $u \in X$ and the injection $i : X_* \hookrightarrow W$ is compact.

As a consequence of Theorem 2.28, we have the following corollary.

Corollary 2.30. *Let X be symmetrically embedded in W and $f : X \to \mathbb{R}$ be a lower semi-continuous functional bounded from below such that (2.19)–(2.23) hold. Then f admits an $(SQPS)_{\inf f}$-sequence converging weakly in X and strongly in W. If in addition $(SQPS)_{\inf f}$ holds, there exists a point $z \in S$ such that $f(z) = \inf f$, $|df|(z) = 0$, $z = z^*$ and $Q_z \geqslant 0$.*

Proof. Let C denote a generic constant that might change from line to line. By Theorem 2.28, f admits an $(SQPS)_{\inf f}$-sequence $(v_h) \subset X$. By construction (v_h) is bounded in X. In fact, with the notation in the proof of Theorem 2.28, there exist a vanishing sequence $(\varepsilon_h) \subset \mathbb{R}^+$ and a bounded sequence $(\xi_h) \subset S$, yielding

$$\|v_h\| \leqslant \|v_h-\xi_h\| + \|\xi_h\| \leqslant \varepsilon_h + \|\mathbb{T}_{\varepsilon_h}\xi_h-\xi_h\| + \|\xi_h\| \leqslant \varepsilon_h + 3\|\xi_h\| \leqslant C.$$

Hence, there exist $v \in X$ and a subsequence of (v_h) that we shall still indicate by (v_h), such that (v_h) weakly converges to v in X. Since X is symmetrically embedded into W, we have that $\|v_h^*\| \leqslant \|v_h\| \leqslant C$ and also, up to a further subsequence, (v_h) converges in W to some $\hat v \in W$. Of course, it is $v = \hat v$. If f satisfies $(SQPS)_{\inf f}$, then there exists a further subsequence, that we still denote by (v_h), which converges to some z in X. By lower semi-continuity, $f(z) = \inf f$. Since $|df|(v_h) \to 0$ and $f(v_h) \to \inf f = f(z)$, by means of Degiovanni and Marzocchi [**12**, Proposition 2.6], it follows that $|df|(z) \leqslant \liminf_h |df|(v_h) = 0$. Since $\|v_h - v_h^*\|_V \to 0$, letting $h \to \infty$ into $\|z-z^*\|_V \leqslant \|z-v_h\|_V + \|v_h-v_h^*\|_V + \|v_h^* - z^*\|_V \leqslant K(C_\Theta+1)\|v_h-z\| + \|v_h-v_h^*\|_V$ yields $z = z^* \in S$, as desired. Since $f(z) = \inf f$ and, by definition, $f(z+t\zeta) \geqslant f(z)$ and $f(z-t\zeta) \geqslant f(z)$ for all $t \in \mathbb{R}$ and $\zeta \in X$, we infer that, for all $w \in X$,

$$Q_z(w) \geqslant \limsup_{\substack{\zeta\to w\\ t\to 0}} \frac{f(z+t\zeta)+f(z-t\zeta)-2f(z)}{t^2} \geqslant 0.$$

This concludes the proof of the corollary. □

These results look particularly useful for applications to PDEs defined on a ball Ω or on $\mathbb{R}^N$, choosing $X = W_0^{1,p}(\Omega)$, $X = S$ or $S = W_0^{1,p}(\Omega, \mathbb{R}^+)$, $V = L^p \cap L^{p^*}(\Omega)$ and $W = L^q(\Omega) \supset V$ with $p < q < p^*$. These functional spaces are compatible with Definition 2.29.

2.7. *Symmetric inf sup principles*

The symmetric version of Ekeland's variational principle allows us to obtain a symmetric minimax type result for C^1 smooth functionals. We refer to Theorem 5.1 in the book [19] by de Figueiredo for a standard statement of the minimax principle, without symmetry, proved through the classical Ekeland's principle. In fact, let $(S, X, V, h, *)$ be according to Definition 2.1. Recall also inequalities (2.1). We assume that $\Theta : (X, \|\cdot\|) \to (S, \|\cdot\|)$ is continuous and also consider the polarization and symmetrization maps as defined over the whole X by setting $u^H = (\Theta(u))^H$ and $u^* = (\Theta(u))^*$, respectively. Let $\psi \in S$ with $\psi^H = \psi$ for all $H \in \mathcal{H}_*$ and introduce the spaces

$$\hat{X} := C([0,1], X), \quad \|\gamma\|_{\hat{X}} := \sup_{t\in[0,1]} \|\gamma(t)\|, \quad \hat{V} := C([0,1], V) \quad \|\gamma\|_{\hat{V}} := \sup_{t\in[0,1]} \|\gamma(t)\|_V,$$
$$\hat{S} := \{\gamma \in C([0,1], X) : \gamma(0) = 0, \quad \gamma(1) = \psi\}.$$

Define $* : \hat{S} \to \hat{V}$, $\gamma \mapsto \gamma^*$ and $h : \hat{S} \times \mathcal{H}_* \to \hat{S}$, $(\gamma, H) \mapsto \gamma^H$ by setting

$$h(\gamma, H)(t) := \gamma(t)^H, \quad \gamma^*(t) := \gamma(t)^*, \quad \forall \gamma \in \hat{S}, \quad \forall H \in \mathcal{H}_*, \ \forall t \in [0,1].$$

Note that, since X and V are Banach spaces, $\hat{X}$ and $\hat{V}$ are Banach spaces, $\hat{S} \subset \hat{X} \subset \hat{V}$, $\hat{S}$ is a closed subset of $\hat{X}$ and $\hat{X}$ is continuously embedded into $\hat{V}$. Furthermore, for all $\gamma \in \hat{S}$ it holds $\gamma^* \in \hat{V}$ and $\gamma^H \in \hat{S}$ since $\gamma^* \in C([0,1], V)$, $\gamma^H \in C([0,1], X)$ and $\gamma^H(0) = \gamma(0)^H = 0$, $\gamma^H(1) = \psi^H = \psi$. With the above definitions, the string $(\hat{S}, \hat{X}, \hat{V}, h, *)$ satisfies the axiomatic properties of Definition 2.1. Let us prove that $\hat{h}$ is a continuous mapping. Given (γ_j, H_j) in $\hat{S} \times \mathcal{H}_*$ such that (γ_j, H_j) converges to (γ_0, H_0) as $j \to \infty$, we have a sequence $(t_j) \subset [0,1]$ converging to a $t_0 \in [0,1]$ as $j \to \infty$, such that

$$\begin{aligned}\|\gamma_j^{H_j} - \gamma_0^{H_0}\|_{\hat{X}} &\leqslant \|\gamma_j(t_j)^{H_j} - \gamma_0(t_j)^{H_j}\| + \|\gamma_0(t_j)^{H_j} - \gamma_0(t_0)^{H_0}\| + \|\gamma_0(t_j)^{H_0} - \gamma_0(t_0)^{H_0}\| \\ &\leqslant C_\Theta\|\gamma_j - \gamma_0\|_{\hat{X}} + \|\gamma_0(t_j)^{H_j} - \gamma_0(t_0)^{H_0}\| + C_\Theta\|\gamma_0(t_j) - \gamma_0(t_0)\|,\end{aligned}$$

yielding the desired conclusion since $\gamma_j \to \gamma$ in $\hat{X}$ as $j \to \infty$, $(\gamma_0(t_j), H_j)$ converges to $(\gamma_0(t_0), H_0)$ and the mapping $h : S \times \mathcal{H}_* \to S$ is continuous. Analogously, given $\gamma \in \hat{S}$, there exists a sequence $(t_m) \subset [0,1]$ converging to $t_0 \in [0,1]$ such that

$$\begin{aligned}\|\gamma^{H_1\ldots H_m} - \gamma^*\|_{\hat{V}} &\leqslant \|\gamma(t_m)^{H_1\ldots H_m} - \gamma(t_0)^{H_1\ldots H_m}\|_V + \|\gamma(t_0)^{H_1\ldots H_m} - \gamma(t_0)^*\|_V \\ &\quad + \|\gamma(t_m)^* - \gamma(t_0)^*\|_V \leqslant \|\gamma(t_0)^{H_1\ldots H_m} - \gamma(t_0)^*\|_V \\ &\quad + 2C_\Theta\|\gamma(t_m) - \gamma(t_0)\|_V,\end{aligned}$$

implying the desired convergence. Also, taken any $\gamma, \eta \in \hat{S}$ and $H \in \mathcal{H}_*$, it holds

$$\|\gamma^H - \eta^H\|_{\hat{V}} = \sup_{t\in[0,1]} \|\gamma(t)^H - \eta(t)^H\|_V \leqslant C_\Theta \sup_{t\in[0,1]} \|\gamma(t) - \eta(t)\|_V \leqslant C_\Theta\|\gamma - \eta\|_{\hat{V}}.$$

Of course, $(\gamma^H)^* = (\gamma^*)^H = \gamma^*$ and $\gamma^{HH} = \gamma^H$ follow immediately from the corresponding property (iii) of Definition 2.1. Now, given a C^1 smooth functional $f : X \to \mathbb{R}$ satisfying

$$f(u^H) \leqslant f(u), \quad \text{for all } u \in X \text{ and } H \in \mathcal{H}_*, \tag{2.27}$$

$$\inf_{\gamma\in\hat{S}} \max_{t\in[0,1]} f(\gamma(t)) > \max\{f(0), f(\psi)\}, \tag{2.28}$$

consider the minimax value

$$c = \inf_{\gamma\in\hat{S}} \max_{t\in[0,1]} f(\gamma(t)),$$

and the functional $\hat{f} : \hat{S} \to \mathbb{R}$, bounded from below in view of (2.28), defined by

$$\hat{f}(\gamma) := \max_{t\in[0,1]} f(\gamma(t)), \quad \text{for all } \gamma \in \hat{S}.$$

Note that $\hat{f}$ is continuous. In fact, let $\bar{\gamma} \in \hat{S}$ and $\varepsilon > 0$. There exists $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ for all $y \in \bar{\gamma}([0,1])$ and all $x \in X$ such that $\|x - y\| \leqslant \delta$. Hence, for all $\gamma \in \hat{S}$ with $\|\gamma - \bar{\gamma}\|_{\hat{X}} \leqslant \delta$,

$$\hat{f}(\gamma) - \hat{f}(\bar{\gamma}) = f(\gamma(\tau)) - \max_{t\in[0,1]} f(\bar{\gamma}(t)) \leqslant |f(\gamma(\tau)) - f(\bar{\gamma}(\tau))|,$$

$\tau \in [0,1]$ being the point where the maximum of $t \mapsto f(\gamma(t))$ is achieved. Then, since $\|\gamma(\tau) - \bar{\gamma}(\tau)\| \leqslant \|\gamma - \bar{\gamma}\|_{\hat{X}} \leqslant \delta$, it follows that $|f(\gamma) - f(\bar{\gamma})| \leqslant \varepsilon$, proving the continuity of $\hat{f}$, after reverting the role of γ and $\hat{\gamma}$. Moreover, due to (2.27), it follows

$$\hat{f}(\gamma^H) = \max_{t\in[0,1]} f(\gamma(t)^H) \leqslant \max_{t\in[0,1]} f(\gamma(t)) = \hat{f}(\gamma), \quad \text{for all } \gamma \in \hat{S}.$$

Then, by applying Theorem 2.7 (with $S' = \hat{S}$ and $\sigma = \rho = \varepsilon > 0$) in place of the standard Ekeland's principle, for every $\varepsilon > 0$ there exists $\gamma_\varepsilon \in \hat{S}$ such that

$$\|\gamma_\varepsilon - \gamma_\varepsilon^*\|_{\hat{V}} < \varepsilon, \quad c \leqslant \hat{f}(\gamma_\varepsilon) \leqslant c + \varepsilon, \quad \hat{f}(\gamma) \geqslant \hat{f}(\gamma_\varepsilon) - \varepsilon\|\gamma - \gamma_\varepsilon\|_{\hat{X}}, \quad \forall \gamma \in \hat{S}.$$

Once these inequalities are reached, by arguing *exactly* as in [**19**, Proof of Theorem 5.1, pp. 37–39], for every $\varepsilon > 0$ there exists a point $u_\varepsilon \in X$ which, by construction, is of the form $u_\varepsilon = \gamma_\varepsilon(t_\varepsilon)$ for some $t_\varepsilon \in [0,1]$, such that

$$\|df(u_\varepsilon)\| \leqslant \varepsilon, \quad c \leqslant f(u_\varepsilon) \leqslant c + \varepsilon.$$

Furthermore, it follows that $\|u_\varepsilon - u_\varepsilon^*\|_V < \varepsilon$, since

$$\|u_\varepsilon - u_\varepsilon^*\|_V = \|\gamma_\varepsilon(t_\varepsilon) - \gamma_\varepsilon(t_\varepsilon)^*\|_V \leqslant \|\gamma_\varepsilon - \gamma_\varepsilon^*\|_{\hat{V}} < \varepsilon.$$

Similar results were obtained in [**30**] without using Ekeland's variational principle.

This kind of achievement is very useful in the study of elliptic equations, especially those set on the entire space $\mathbb{R}^N$ where the addition of almost symmetry information yields a compactifying effect, through compact embeddings of spaces of symmetric functions X_{sym} of X into X.

3. *Some applications*

In this section, we highlight possible applications of the abstract symmetric versions of the variational principles in the framework of PDEs, fixed point theory and geometric properties of Banach spaces.

3.1. *Calculus of variations*

In this section, we consider two applications of the symmetric principles to PDEs.

3.1.1. *A quasi-linear example.* Let $\Omega = B$ be the unit ball in $\mathbb{R}^N$ ($N \geqslant 1$), $1 < p < \infty$ and define the functional $f : W_0^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by setting

$$f(u) = \int_\Omega \mathcal{L}(u, |Du|), \tag{3.1}$$

where $\mathcal{L}$ is an integrand of class C^1 and, for $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$,

$$\mathcal{L}(s, |\xi|) \geqslant 0. \tag{3.2}$$

Assume that u belongs to $\mathrm{dom}(f)$ whenever $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. The functions $\mathcal{L}_s$ and $\mathcal{L}_\xi$ are the derivatives of $\mathcal{L}$ with respect to the variables s and ξ. We assume that there exist

$\alpha, \beta, \gamma \in C(\mathbb{R})$ and real numbers $a, b \in \mathbb{R}$ such that the following conditions hold:

$$|\mathcal{L}(s, |\xi|)| \leqslant \alpha(|s|)|\xi|^p + b|\xi|^p + a, \tag{3.3}$$

$$|\mathcal{L}_s(s, |\xi|)| \leqslant \beta(|s|)|\xi|^p, \quad |\mathcal{L}_\xi(s, |\xi|)| \leqslant \gamma(|s|)|\xi|^{p-1} + b|\xi|^{p-1} + a, \tag{3.4}$$

for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. We write the growth assumptions in such a fashion, since in the particular case with $\beta = \gamma = 0$, conditions (3.2)–(3.4) reduce to [**15**, Assumptions (4.12)–(4.14)] as stated by Ekeland. Now, since, in the general case where β and γ are unbounded, $\mathcal{L}_s(u, |Du|)$ and $\mathcal{L}_\xi(u, |Du|)$ are not in $L^1_{\mathrm{loc}}(B)$ for a given function $u \in W^{1,p}_0(\Omega)$, the Euler–Lagrange equation associated with f cannot be given, at least a priori, a distributional sense. To overcome this situation, in [**25**], for every $u \in W^{1,p}_0(\Omega)$ the following vector space, dense in $W^{1,p}_0(\Omega)$, was used:

$$V_u = \{v \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega) : \ u \in L^\infty(\{x \in \Omega : \ v(x) \neq 0\})\}. \tag{3.5}$$

The following proposition can be obtained arguing as in [**25**, Proposition 4.5] and provides a link between the weak slope and directional derivatives of f along a direction $v \in V_u$.

PROPOSITION 3.1. *Under assumptions* (3.2)–(3.4), *for every* $u \in \mathrm{dom}(f)$, *we have*

$$|df|(u) \geqslant \sup_{\substack{v \in V_u \\ \|v\|_{1,p} \leqslant 1}} \left[\int_\Omega \mathcal{L}_\xi(u, |Du|) \cdot Dv + \int_\Omega \mathcal{L}_s(u, |Du|)v\right].$$

As a consequence of Proposition 3.1 and Theorem 2.8, we have the following theorem.

THEOREM 3.2. *Assume that conditions* (3.2)–(3.4) *hold and* $\mathcal{L}(-s, |\xi|) \leqslant \mathcal{L}(s, |\xi|)$ *for all* $s \leqslant 0$. *Then, for any* $\varepsilon > 0$, *there exist* $u_\varepsilon \in W^{1,p}_0(\Omega)$ *and* $w_\varepsilon \in W^{-1,p'}(\Omega)$ *such that*

$$\langle w_\varepsilon, v\rangle = \int_\Omega \mathcal{L}_\xi(u_\varepsilon, |Du_\varepsilon|) \cdot Dv + \int_\Omega \mathcal{L}_s(u_\varepsilon, |Du_\varepsilon|)v \quad \forall v \in V_{u_\varepsilon}, \tag{3.6}$$

as well as

$$\|w_\varepsilon\|_{W^{-1,p'}(\Omega)} \leqslant \varepsilon \quad \text{and} \quad \|u_\varepsilon - u^*_\varepsilon\|_{L^p(\Omega) \cap L^{p^*}(\Omega)} < \varepsilon.$$

Proof. The functional f in formula (3.1) is proper, bounded from below and lower semi-continuous by means of condition (3.2) and Fatou's lemma. Moreover, the assumptions of Theorem 2.8 are satisfied with $X = W^{1,p}_0(\Omega)$, $S = W^{1,p}_0(\Omega, \mathbb{R}^+)$, $V = L^p(\Omega) \cap L^{p^*}(\Omega)$, $\xi = |u|$ and where u^H, u^* for $u \in S$ and $u^* = |u|^*$ for $u \in X$ denote the polarization and symmetrization, respectively (see Sections 2.1.1–2.1.2). Assumption (2.3) holds with equal sign by the radial structure of the integrand, as it can be verified by direct computation. The assertion follows by Theorem 2.8 (recall also Remark 2.23), Proposition 3.1 and the Hahn–Banach theorem, taking into account the density of V_{u_ε} in $W^{1,p}_0(\Omega)$. □

In many cases, one recovers the fact that the solution u_ε of equation (3.6) is actually meant in the sense of distributions, by suitably enlarging the class of admissible test functions; see, for example, [**25**, Theorem 4.10 and Lemma 4.6]. Theorem 3.2 could be seen as a nonsmooth symmetric version of Ekeland [**15**, Proposition 4.3(a)]. In fact, under the above assumptions our functional is merely lower semi-continuous, while the functional of Ekeland [**15**, Proposition 4.3(a)] is of class C^1. Furthermore, the symmetry featured in Theorem 3.2 can be obtained via Theorem 2.8 due to the structure $\mathcal{L}(s, |\xi|)$ yielding (2.3), in place of the more general form $\mathcal{L}(x, s, \xi)$, admissible in [**15**]. Theorem 3.2 is new even in the particular case $\beta = \gamma = 0$. We stress that a constrained version of Theorem 3.2 could also be provided, yielding a nonsmooth symmetric counterpart of Ekeland [**15**, Proposition 4.3(b)].

REMARK 3.3. It should be highlighted that the strength of Theorem 3.2 is related to the fact that we are *not* assuming that $\xi \mapsto j(x, s, \xi)$ is convex. Should one additionally consider this assumption, it is then often the case that functionals satisfying (2.3), fulfil in turn the corresponding symmetrization inequality $f(u^*) \leqslant f(u)$. In such a case, starting from a minimizing sequence (u_h), one has that (u_h^*) is a minimizing sequence too and it is then easy to find a further almost symmetric minimizing sequence (v_h). Without the convexity of $\xi \mapsto j(x, s, \xi)$, to the author's knowledge, no symmetrization inequality is available and thus Theorem 3.2 has a rather significant impact on PDEs and problems of Calculus of Variations which are set on a symmetric domain.

3.1.2. *A semi-linear example.* Let us now briefly discuss another example where the second-order condition related to $w \mapsto Q_u(w)$ is also involved, namely, the inferior limit in formula (2.18), in a C^1, but not C^2, framework. In [**2**], Bartsch and Degiovanni showed that, in some concrete cases of interest in the theory of PDEs, although it is often not possible to compute the values of $Q_u(w)$, it is possible to compute a greater quantity. For instance, if f is of class C^1, then (see [**2**, Remark 4.4]), for every $w \in X$,

$$Q_u(w) \leqslant \limsup_{\substack{(\tau,\vartheta)\to(0,0)\\ z\to u\\ \zeta\to w}} \frac{f'(z+\tau\zeta)\zeta - f'(z+\vartheta\zeta)\zeta}{\tau - \vartheta},$$

the right-hand side being easier to estimate, in some cases [**2**, Propositions 4.5]. For instance, now let $\Omega = B$ be the unit ball in $\mathbb{R}^3$, the three-dimensional case being considered just for simplicity. Let also $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and assume that there exist $a_1, a_2 \in \mathbb{R}$, $b \in \mathbb{R}$ and $2 < p \leqslant 6$ such that, for all $s, t \in \mathbb{R}$, it holds

$$|g(s)| \leqslant a_1 + b|s|^{p-1} \quad \text{and} \quad g(-s) = -g(s),$$
$$(g(s) - g(t))(s - t) \geqslant -(a_2 + b|s|^{p-2} + b|t|^{p-2})(s - t)^2.$$

Then, for all $s \in \mathbb{R}$, define a measurable function $\underline{D}_s g$ by setting

$$\underline{D}_s g(s) := \liminf_{\substack{(t,\tau)\to(0,0)\\ t,\tau\in\mathbb{Q}}} \frac{g(s+t) - g(s+\tau)}{t - \tau}.$$

Let $G(s) = \int_0^s g(t)dt$ and consider the C^1 functional $f : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega |Du|^2 - \int_\Omega G(u).$$

In light of Bartsch and Degiovanni [**2**, Proposition 6.1], it holds

$$Q_u(w) \leqslant \int_\Omega |Dw|^2 - \int_\Omega \underline{D}_s g(u) w^2 < +\infty, \quad \forall u, w \in H_0^1(\Omega). \tag{3.7}$$

Therefore, combining Theorem 2.28 with the above setting yields the following theorem.

THEOREM 3.4. *Assume that f is bounded from below and admits a bounded minimizing sequence. Then f has a minimizing sequence $(u_h) \subset H_0^1(\Omega)$ and a sequence $(\psi_h) \subset H^{-1}(\Omega)$*

such that

$$\lim_h \|u_h - u_h^*\|_{L^2(\Omega)\cap L^{2^*}(\Omega)} = 0,$$
$$\int_\Omega Du_h D\varphi = \int_\Omega g(u_h)\varphi + \langle \psi_h, \varphi\rangle, \quad \forall \varphi \in H_0^1(\Omega), \quad \lim_h \|\psi_h\|_{H^{-1}} = 0,$$
$$\liminf_h \left[\int_\Omega |Dw|^2 - \int_\Omega \underline{D}_s g(u_h) w^2\right] \geqslant 0, \quad \forall w \in H_0^1(\Omega).$$

Proof. Based upon the above remarks, the assertion follows by Theorem 2.28 by choosing $X = H_0^1(\Omega)$, $S = H_0^1(\Omega, \mathbb{R}^+)$, $V = L^2(\Omega)\cap L^{2^*}(\Omega)$, since $f(u^H) = f(u)$ for all $u \in S$ and $H \in \mathcal{H}_*$, as well as $f(|u|) = f(u)$ for all $u \in X$ and $\|u^H\|_{H_0^1(\Omega)} = \|u\|_{H_0^1(\Omega)}$ for all $u \in S$ and $H \in \mathcal{H}_*$. □

3.2. *Fixed points*

The following is a symmetric version of the so-called Caristi Fixed Point Theorem [**6**], which was also proved by Ekeland via his principle in [**16**].

Theorem 3.5 (Symmetric Caristi Fixed Point Theorem). *Let X be a Banach space and $F : X \to X$ be a map such that*

$$\|F(u) - u\| \leqslant f(u) - f(F(u)), \quad \textit{for all } u \in X,$$

where $f : X \to \mathbb{R}$ is a lower semi-continuous function, bounded from below, satisfying (2.3) and such that, for all $u \in X$, there exists $\xi \in S$ with $f(\xi) \leqslant f(u)$. Then, for all $\varepsilon \in (0,1)$, there exists a fixed point $\xi_\varepsilon \in X$ of F such that $\|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon$.

Proof. By virtue of Theorem 2.8 with $\sigma = \rho = \varepsilon > 0$, for every $\varepsilon \in (0,1)$, there exists an element $\xi_\varepsilon \in X$ such that $\|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon$, and

$$f(w) \geqslant f(\xi_\varepsilon) - \varepsilon\|w - \xi_\varepsilon\| \quad \text{for all } w \in X.$$

In particular, choosing $w = F(\xi_\varepsilon)$ and using the assumption, we get

$$\|F(\xi_\varepsilon) - \xi_\varepsilon\| \leqslant f(\xi_\varepsilon) - f(F(\xi_\varepsilon)) \leqslant \varepsilon\|F(\xi_\varepsilon) - \xi_\varepsilon\|,$$

which yields $F(\xi_\varepsilon) = \xi_\varepsilon$, concluding the proof. □

Let Ω be either a ball in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$ and take $1 < p < \infty$.

Corollary 3.6. *Let $F : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ be a map such that*

$$\|F(u) - u\|_{1,p} \leqslant f(u) - f(F(u)), \quad \textit{for all } u \in W_0^{1,p}(\Omega),$$

where $f : W_0^{1,p}(\Omega) \to \mathbb{R}$ is a lower semi-continuous function bounded from below such that

$$f(|u|) \leqslant f(u) \quad \textit{for all } u \in W_0^{1,p}(\Omega), \quad f(u^H) \leqslant f(u) \quad \textit{for all } u \in W_0^{1,p}(\Omega, \mathbb{R}^+).$$

Then, for all $\varepsilon \in (0,1)$, there is a fixed point $\xi_\varepsilon \in W_0^{1,p}(\Omega)$ of F with $\|\xi_\varepsilon - \xi_\varepsilon^*\|_{L^p\cap L^{p^*}(\Omega)} < \varepsilon$.

Proof. Theorem 3.5 is applied with $X = W_0^{1,p}(\Omega)$, $S = W_0^{1,p}(\Omega, \mathbb{R}^+)$ and $V = L^p \cap L^{p^*}(\Omega)$. As pointed out on Section 2.1, if u^H is the polarization of positive functions on $\mathbb{R}^N$ and $*$ is the Schwarz symmetrization, the framework of Definition 2.1 is satisfied. □

Let Ω be either a ball in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$ and take $1 < p < \infty$.

Corollary 3.7. *Let $F : L^p(\Omega) \to L^p(\Omega)$ be a map such that*

$$\|F(u) - u\|_p \leqslant f(u) - f(F(u)), \quad \text{for all } u \in L^p(\Omega),$$

where $f : L^p(\Omega) \to \mathbb{R}$ is a lower semi-continuous function bounded from below such that

$$f(|u|) \leqslant f(u) \quad \text{for all } u \in L^p(\Omega), \quad f(u^H) \leqslant f(u) \quad \text{for all } u \in L^p(\Omega, \mathbb{R}^+).$$

Then, for all $\varepsilon \in (0, 1)$, there is a fixed point $\xi_\varepsilon \in L^p(\Omega)$ of F with $\|\xi_\varepsilon - \xi_\varepsilon^*\|_{L^p(\Omega)} < \varepsilon$.

Proof. Theorem 3.5 is applied with $X = V = L^p(\Omega)$ and $S = L^p(\Omega, \mathbb{R}^+)$. □

We conclude the section with a symmetric version of a fixed point theorem due to Clarke [**8**] and also proved by Ekeland via his principle [**16**].

Theorem 3.8. *Let $(X, \|\cdot\|_V)$ be a Banach space, and $F : (X, \|\cdot\|_V) \to (X, \|\cdot\|_V)$ be continuous, and assume that there exists $0 < \sigma < 1$ such that*

$$\forall u \in X \ \exists t \in (0, 1] : \ \|F(tF(u) + (1 - t)u) - F(u)\|_V \leqslant \sigma t\|F(u) - u\|_V. \tag{3.8}$$

Assume that $F(S) \subset S$, $F(u^H) = F(u)^H$ for all $H \in \mathcal{H}_*$ and $u \in S$, and that, for every $u \in X$, there exists $\xi \in S$ such that $\|\xi - F(\xi)\|_V \leqslant \|u - F(u)\|_V$. Then, for any $\varepsilon \in (0, 1 - \sigma)$, there exists a fixed point $\xi_\varepsilon \in X$ for F such that $\|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon$.

Proof. It is sufficient to argue essentially as in the proof of Ekeland [**16**, Theorem 3] on the function $f : X \to \mathbb{R}$ defined by $f(u) := \|u - F(u)\|_V$ observing that, by assumption and by (v) of Definition 2.1, it holds $f(u^H) = \|u^H - F(u^H)\|_V = \|u^H - F(u)^H\|_V \leqslant \|u - F(u)\|_V = f(u)$ for all $H \in \mathcal{H}_*$ and $u \in S$. Moreover, for all $u \in X$ there is $\xi \in S$ such that $f(\xi) \leqslant f(u)$. Applying Theorem 2.8 in place of Ekeland's principle, the assertion follows. □

Let Ω be either a ball in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$ and take $1 < p < \infty$.

Corollary 3.9. *Let $F : L^p(\Omega) \to L^p(\Omega)$ be a map such that (3.8) holds, $F(u) \geqslant 0$ for all $u \in L^p(\Omega, \mathbb{R}^+)$, $F(u^H) = F(u)^H$ for all $H \in \mathcal{H}_*$ and $u \in L^p(\Omega, \mathbb{R}^+)$, and that $F(|u|) = |F(u)|$ for all $u \in L^p(\Omega)$. Then, for every $\varepsilon \in (0, 1 - \sigma)$, there is a fixed point $\xi_\varepsilon \in L^p(\Omega)$ for F such that $\|\xi_\varepsilon - \xi_\varepsilon^*\|_{L^p(\Omega)} < \varepsilon$.*

Proof. Apply Theorem 3.8 with the choice $X = V = L^p(\Omega)$, $S = L^p(\Omega, \mathbb{R}^+)$. Note that, for all $u \in X$, it holds $\||u| - F(|u|)\|_{L^p(\Omega)} = \||u| - |F(u)|\|_{L^p(\Omega)} \leqslant \|u - F(u)\|_{L^p(\Omega)}$, in the notation of the proof of Theorem 3.8. □

3.3. *Drops and flower petals*

As a by-product of the symmetric variational principle, Theorem 2.7, we obtain symmetric versions of the Daneš Drop Theorem [**10**] and of the Flower Petal Theorem [**26**]. In the particular case where h and $*$ are the identity maps and $S = X = V$, the statements reduce to the classical formulation. Possible applications of the statements to some meaningful concrete situations have not yet been investigated.

DEFINITION 3.10. Let X be a Banach space, $B \subset X$ be convex and $x \in X$'. We say that

$$\mathrm{Drop}(x, B) := \bigcup_{y\in B,\ t\in[0,1]} x + t(y - x),$$

is the drop associated with x and B. If $x_0, x_1 \in X$ and $\varepsilon > 0$, then we say that

$$\mathrm{Petal}_\varepsilon(x_0, x_1) := \big\{y \in X : \varepsilon\|y - x_0\| + \|y - x_1\| \leqslant \|x_0 - x_1\|\big\}$$

is the petal associated with ε and $x_0, x_1 \in X$.

Note that, for all $\varepsilon \in (0, 1)$ and $x_0, x_1 \in X$, it always holds

$$B_{((1-\varepsilon)/(1+\varepsilon))\|x_0-x_1\|}(x_1) \subset \mathrm{Petal}_\varepsilon(x_0, x_1),$$
$$\mathrm{Drop}(x_0, B_{((1-\varepsilon)/(1+\varepsilon))\|x_0-x_1\|}(x_1)) \subset \mathrm{Petal}_\varepsilon(x_0, x_1),$$

so that each petal contains a suitable ball as well as a drop of a suitable ball.

Here is a symmetric version of the so-called *Drop* Theorem due to Daneš [**10**].

THEOREM 3.11 (Symmetric Drop Theorem). *Let $(X, \|\cdot\|_V)$ be a Banach space, B, C be nonempty closed subsets of S, with $B \subset X_{\mathcal{H}_*}$ convex, and $d(B, C) > 0$. Moreover, let $x \in C$ such that $S' := \mathrm{Drop}(x, B) \cap C$ is closed and $h(S') \subset S'$, $*(S') \subset V$. Then, for all $\varepsilon > 0$ small, there exists $\xi_\varepsilon \in \mathrm{Drop}(x, B) \cap C$ such that*

$$\mathrm{Drop}(\xi_\varepsilon, B) \cap C = \{\xi_\varepsilon\} \quad \textit{and} \quad \|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon.$$

Proof. By Remark 2.3, $(S', X, V, h, *)$ satisfies (i)–(v) of Definition 2.1 and Proposition 2.2. Moreover, S' is closed. Define a continuous function $f : S' \to \mathbb{R}^+$ by setting

$$f(u) := \inf_{\zeta\in B} \|u - \zeta\|_V, \quad \text{for all } u \in S'.$$

Observe that, since $B \subset X_{\mathcal{H}_*}$, for all $u \in S'$ and any $H \in \mathcal{H}_*$, we have

$$f(u^H) = \inf_{\zeta\in B} \|u^H - \zeta\|_V = \inf_{\zeta\in B} \|u^H - \zeta^H\|_V \leqslant \inf_{\zeta\in B} \|u - \zeta\|_V = f(u),$$

in light of (v) of Definition 2.1. Now let $\varepsilon_0 > 0$ be fixed sufficiently small that $\varepsilon_0 \operatorname{diam}(B) < (1 - \varepsilon_0)d(B, C)$. In turn, for every $\varepsilon \in (0, \varepsilon_0]$, by applying Theorem 2.7 with $\rho = \sigma = \varepsilon$, we find an element $\xi_\varepsilon \in S'$ such that $\|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon$ and

$$\inf_{\zeta\in B} \|w - \zeta\|_V > \inf_{\zeta\in B} \|\xi_\varepsilon - \zeta\|_V - \varepsilon\|w - \xi_\varepsilon\|_V, \quad \forall w \in S \setminus \{\xi_\varepsilon\}. \tag{3.9}$$

To prove the assertion, we argue by contradiction, assuming that

$$\mathrm{Drop}(\xi_\varepsilon, B) \cap (\mathrm{Drop}(x, B) \cap C) \neq \{\xi_\varepsilon\}.$$

Then we find $\tau \in [0, 1]$, $\tau \neq 1$, and $\eta \in B$ such that $\hat{w} = (1 - \tau)\eta + \tau\xi_\varepsilon \in S' \setminus \{\xi_\varepsilon\}$. In turn, from formula (3.9) evaluated at $\hat{w}$, and since B is convex, we infer

$$\inf_{\zeta\in B} \|\xi_\varepsilon - \zeta\|_V < \tau \inf_{\zeta\in B} \|\xi_\varepsilon - \zeta\|_V + (1 - \tau) \inf_{\zeta\in B} \|\eta - \zeta\|_V + \varepsilon(1 - \tau)\|\eta - \xi_\varepsilon\|_V,$$

namely (recall that $0 \leqslant \tau < 1$) for every $\zeta \in B$ it holds

$$\inf_{\zeta\in B} \|\xi_\varepsilon - \zeta\|_V < \varepsilon\|\eta - \xi_\varepsilon\|_V \leqslant \varepsilon \operatorname{diam}(B) + \varepsilon\|\zeta - \xi_\varepsilon\|_V.$$

Therefore, taking the infimum over $\zeta \in B$, and since $\varepsilon \in (0, \varepsilon_0]$, we conclude that

$$(1 - \varepsilon_0)d(B, C) \leqslant (1 - \varepsilon) \inf_{\zeta\in B} \|\xi_\varepsilon - \zeta\|_V \leqslant \varepsilon \operatorname{diam}(B) \leqslant \varepsilon_0 \operatorname{diam}(B) < (1 - \varepsilon_0)d(B, C),$$

which is a contradiction. Hence, $\mathrm{Drop}(\xi_\varepsilon, B) \cap (\mathrm{Drop}(x, B) \cap C) = \{\xi_\varepsilon\}$. By the inclusion $\mathrm{Drop}(\xi_\varepsilon, B) \subset \mathrm{Drop}(x, B)$ we get $\mathrm{Drop}(\xi_\varepsilon, B) \cap C = \{\xi_\varepsilon\}$, concluding the proof. □

Now let Ω be either the unit ball in $\mathbb{R}^N$ or $\mathbb{R}^N$ and $1 < p < \infty$. We denote by $L^p_r(\Omega, \mathbb{R}^+)$ the set of radially symmetric elements of $L^p(\Omega, \mathbb{R}^+)$, that is, $u^* = u$, ($*$ being the Schwarz symmetrization) being equivalent to $u^H = u$ for any $H \in \mathcal{H}_*$.

Corollary 3.12 (Symmetric Drop Theorem in L^p-spaces). *Let C be a nonempty closed subset of $(L^p(\Omega, \mathbb{R}^+), \|\cdot\|_{L^p(\Omega)})$ and B be a unit ball in $L^p_r(\Omega, \mathbb{R}^+)$ with $d(B, C) > 0$. Let $u \in C$ be such that*

$$\forall v \in \mathrm{Drop}(u, B) \cap C, \quad \forall H \in \mathcal{H}_* : \ v^H \in \mathrm{Drop}(u, B) \cap C.$$

Then, for all $\varepsilon > 0$ small, there exists $\xi_\varepsilon \in \mathrm{Drop}(u, B) \cap C$ such that

$$\mathrm{Drop}(\xi_\varepsilon, B) \cap C = \{\xi_\varepsilon\} \quad \textit{and} \quad \|\xi_\varepsilon - \xi_\varepsilon^*\|_{L^p(\Omega)} < \varepsilon.$$

Proof. By assumption, S' is compatible with Definition 2.1. Apply Theorem 3.11 with $X = V = L^p(\Omega)$, $S = L^p(\Omega, \mathbb{R}^+)$, $S' = \mathrm{Drop}(u, B) \cap C$. Since $B \subset L^p_r(\Omega, \mathbb{R}^+)$, $u^* = u$ for all $u \in B$ and thus $u^H = u$ for all $H \in \mathcal{H}_*$. Hence, B is a convex subset of $X_{\mathcal{H}_*}$. □

Next, we state a symmetric version of the *Petal Flower* Theorem obtained by Penot [**26**].

Theorem 3.13 (Symmetric Petal Flower Theorem). *Let $(X, \|\cdot\|_V)$ be a Banach space and $S' = C$ be a closed subset of S such that*

$$\forall v \in C, \quad \forall H \in \mathcal{H}_* : \ v^H \in C.$$

Assume that $x \in C$, $y \in S \setminus C$ with $x^H = x$ and $y^H = y$ for any $H \in \mathcal{H}_*$ and

$$\|x - y\|_V \leqslant d(y, C) + \varepsilon^2 \quad \textit{for some } \varepsilon > 0. \tag{3.10}$$

Then there exists a point $\xi_\varepsilon \in \mathrm{Petal}_\varepsilon(x, y) \cap C$ such that

$$\mathrm{Petal}_\varepsilon(\xi_\varepsilon, y) \cap C = \{\xi_\varepsilon\} \quad \textit{and} \quad \|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon.$$

Proof. By Remark 2.3, $(S', X, V, h, *)$ satisfies (i)–(v) of Definition 2.1 and Proposition 2.2. Moreover, S' is closed. Define the continuous map $f : S' \to \mathbb{R}^+$ by setting $f(u) := \|u - y\|_V$ for all $u \in S'$. Since $y^H = y$ for any $H \in \mathcal{H}_*$, we have

$$f(u^H) = \|u^H - y\|_V = \|u^H - y^H\|_V \leqslant \|u - y\|_V = f(u), \quad \text{for } u \in S' \text{ and } H \in \mathcal{H}_*.$$

Then, by Theorem 2.13 and Remark 2.14, with the choice $\rho = \sigma = \varepsilon$, since (3.10) rephrases as $f(x) \leqslant \inf_{S'} f + \varepsilon^2$, there exists $\xi_\varepsilon \in C$ such that $\|\xi_\varepsilon - \xi_\varepsilon^*\|_V < \varepsilon$,

$$\varepsilon\|w - \xi_\varepsilon\|_V + \|w - y\|_V > \|\xi_\varepsilon - y\|_V, \quad \forall w \in C \setminus \{\xi_\varepsilon\},$$

and $\varepsilon\|\xi_\varepsilon - \mathbb{T}_\varepsilon x\|_V + \|\xi_\varepsilon - y\|_V \leqslant \|x - y\|_V$. As $\mathbb{T}_\varepsilon x = x$, this means $\xi_\varepsilon \in \mathrm{Petal}_\varepsilon(x, y) \cap C$ and $w \notin \mathrm{Petal}_\varepsilon(\xi_\varepsilon, y)$ for all $w \in C \setminus \{\xi_\varepsilon\}$, that is, $\mathrm{Petal}_\varepsilon(\xi_\varepsilon, y) \cap C = \{\xi_\varepsilon\}$. □

Now let Ω be either the unit ball in $\mathbb{R}^N$ or the whole $\mathbb{R}^N$ and take $1 \leqslant p < \infty$.

Corollary 3.14 (Symmetric Petal Flower Theorem in L^p-spaces). *Let C be a closed subset of $(L^p(\Omega, \mathbb{R}^+), \|\cdot\|_{L^p(\Omega)})$, $u \in C$, $v \in L^p(\Omega, \mathbb{R}^+) \setminus C$ with $u^H = u$ and $v^H = v$ for any*

$H \in \mathcal{H}_*$, $\|u - v\|_{L^p(\Omega)} \leqslant d(v, C) + \varepsilon^2$ *for some* $\varepsilon > 0$. *Assume in addition that*

$$\forall v \in C, \quad \forall H \in \mathcal{H}_* : \; v^H \in C.$$

Then there exists $\xi_\varepsilon \in \mathrm{Petal}_\varepsilon(u, v) \cap C$ *with* $\mathrm{Petal}_\varepsilon(\xi_\varepsilon, v) \cap C = \{\xi_\varepsilon\}$ *and* $\|\xi_\varepsilon - \xi_\varepsilon^*\|_{L^p(\Omega)} < \varepsilon$.

Proof. Apply Theorem 3.13, with the choice $X = V = L^p(\Omega)$ and $S = L^p(\Omega, \mathbb{R}^+)$. □

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