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A MULTIPLICITY RESULT FOR PERTURBED SYMMETRIC QUASILINEAR ELLIPTIC SYSTEMS

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Abstract. By means of nonsmooth critical-point theory, we prove existence of infinitely many solutions $(u^m) \subseteq H^1_0(\Omega, \mathbb{R}^N)$ for a class of perturbed $\mathbb{Z}_2$−symmetric elliptic systems.

1. Introduction

In critical-point theory, an open problem concerning existence is the role of symmetry in obtaining multiple critical points for even functionals.

Around 1980, the semilinear scalar problem

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}(x)D_i u) = g(x,u) + \varphi & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

with g superlinear and odd in u and $\varphi \in L^2(\Omega)$, has been the object of a very careful analysis by A. Bahri and H. Berestycki in [3], M. Struwe in [23], G-C. Dong and S. Li in [14] and by P. H. Rabinowitz in [19] via techniques of classical critical-point theory. Around 1990, A. Bahri and P. L. Lions in [4, 5] improved the previous results via a Morse-index-type technique.

Later on, since 1994, several efforts have been devoted to studying existence for quasilinear scalar problems of the type

$$\begin{cases} -\sum\limits_{i,j=1}^{n} D_j(a_{ij}(x,u)D_i u) + \frac{1}{2}\sum\limits_{i,j=1}^{n} D_s a_{ij}(x,u)D_i u D_j u = g(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases}$$

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We refer the reader to [2, 7, 8, 9, 24] and to [1, 18, 22] for a more general setting. In this case the associated functional $f : H_0^1(\Omega) \to \mathbb{R}$ given by

$$f(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n a_{ij}(x,u) D_i u D_j u \, dx - \int_\Omega G(x,u)\, dx$$

is not even locally Lipschitz unless the a_{ij}'s do not depend on u or $n = 1$.

Consequently, techniques of nonsmooth critical-point theory have to be applied. We refer to [10, 13, 15, 17] for the abstract theory and in particular to [9] for the main results we shall need in the following.

It seems now natural to ask whether some existence results for perturbed even functionals still hold in a quasilinear setting, both scalar ($N = 1$) and vectorial ($N \geq 2$).

In [21] one of the authors has recently proved that diagonal quasilinear elliptic systems of the type ($k = 1, \ldots, N$)

$$-\sum_{i,j=1}^n D_j(a_{ij}^k(x,u) D_i u_k) + \frac{1}{2} \sum_{i,j=1}^n \sum_{h=1}^N D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = D_{s_k} G(x,u), \tag{1.2}$$

in Ω, possess a sequence (u^m) of weak solutions in $H_0^1(\Omega, \mathbb{R}^N)$ under suitable assumptions, including symmetry, on coefficients a_{ij}^h and G. In order to prove this result, we looked for critical points of the functional $f_0 : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$f_0(u) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx\,. \tag{1.3}$$

In this paper we want to investigate the effects of destroying the symmetry of system (1.2) and show that for each $\varphi \in L^2(\Omega, \mathbb{R}^N)$ the perturbed problem

$$-\sum_{i,j=1}^n D_j(a_{ij}^k(x,u) D_i u_k) + \tfrac{1}{2} \sum_{i,j=1}^n \sum_{h=1}^N D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = D_{s_k} G(x,u) + \varphi_k \tag{1.4}$$

in Ω, still has infinitely many weak solutions. Of course, to this aim, we shall study the associated functional

$$f(u) = \tfrac{1}{2} \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx - \int_\Omega \varphi \cdot u \, dx. \tag{1.5}$$

In the following, Ω will denote an open and bounded subset of $\mathbb{R}^n$. In order to adapt the perturbation argument of [19], we shall consider the following assumptions:

(a) the matrix $\big(a_{ij}^h(x,s)\big)$ is measurable in x for each $s \in \mathbb{R}^N$ and of class C^1 in s for almost every $x \in \Omega$ with $a_{ij}^h(x,s) = a_{ji}^h(x,s)$. Moreover, there exist $\nu > 0$ and $C > 0$ such that

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x,s)\xi_i^h \xi_j^h \geq \nu |\xi|^2, \quad \big|a_{ij}^h(x,s)\big| \leq C, \quad \big|D_s a_{ij}^h(x,s)\big| \leq C, \tag{1.6}$$

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} s \cdot D_s a_{ij}^h(x,s)\xi_i^h \xi_j^h \geq 0\,, \tag{1.7}$$

for almost every $x \in \Omega$ and for all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$;

(b) (if $N \geq 2$) there exists a bounded Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} \big(\frac{1}{2} D_s a_{ij}^h(x,s) \cdot \exp_\sigma(r,s) + a_{ij}^h(x,s) D_{s_h}(\exp_\sigma(r,s))_h\big)\xi_i^h \xi_j^h \leq 0\,, \tag{1.8}$$

for almost every $x \in \Omega$, for all $\xi \in \mathbb{R}^{nN}$, $\sigma \in \{-1,1\}^N$ and $r, s \in \mathbb{R}^N$, where $(\exp_\sigma(r,s))_i := \sigma_i \exp[\sigma_i(\psi(r_i) - \psi(s_i))]$, for each $i = 1, \dots, N$.

(c) the function $G(x,s)$ is measurable in x for all $s \in \mathbb{R}^N$, of class C^1 in s for almost every $x \in \Omega$ with $G(x,0) = 0$ and $g(x,\cdot)$ denotes the gradient of G with respect to s.

(d) there exist $q > 2$ and $R > 0$ such that

$$|s| \geq R \Longrightarrow 0 < qG(x,s) \leq s \cdot g(x,s)\,, \tag{1.9}$$

for almost every $x \in \Omega$ and all $s \in \mathbb{R}^N$;

(e) there exists $\gamma \in (0, q-2)$ such that

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} s \cdot D_s a_{ij}^h(x,s)\xi_i^h \xi_j^h \leq \gamma \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x,s)\xi_i^h \xi_j^h\,, \tag{1.10}$$

for almost every $x \in \Omega$ and for all $s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^{nN}$.

Under the previous assumptions, the following is our main result.

Theorem 1.1. *Assume that there exists* $\sigma \in (1, \frac{qn+(q-1)(n+2)}{qn+(q-1)(n-2)})$ *such that*

$$|g(x,s)| \leq a + b|s|^\sigma, \tag{1.11}$$

with $a, b \in \mathbb{R}$ *and that for almost every* $x \in \Omega$ *and for each* $s \in \mathbb{R}^N$, $a_{ij}^h(x,-s) = a_{ij}^h(x,s)$, $g(x,-s) = -g(x,s)$. *Then there exists a sequence* $(u^m) \subseteq H_0^1(\Omega, \mathbb{R}^N)$ *of solutions to the system*

$$-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u)D_i u_k) + \frac{1}{2}\sum_{i,j=1}^{n}\sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = D_{s_k} G(x,u) + \varphi_k$$

in Ω, *such that* $\lim_m f(u^m) = +\infty$.

This is clearly an extension of the results of [3, 14, 19, 23] to the quasilinear case, both scalar ($N = 1$) and vectorial ($N \geq 2$).

Let us point out that in the case $N = 1$ a stronger version of the previous result can be proven. Indeed, we may completely drop assumption (b) and replace Lemma 4.1 with [9, Lemma 2.2.4].

To our knowledge, in the case $N > 1$ only very few multiplicity results have been obtained so far via nonsmooth critical-point theory (see [2, 21, 24]).

In [2], for coefficients of type $a_{ij}^{hk} = \alpha_{ij}\delta^{hk}$, a new technical condition was introduced to be compared with hypothesis (b). They assume that there exist $K > 0$ and an increasing bounded Lipschitz function $\psi : [0, +\infty) \to [0, +\infty)$, with $\psi(0) = 0$, ψ' nonincreasing, $\psi(t) \to K$ as $t \to +\infty$ and such that

$$\sum_{\substack{i,j=1\\k=1}}^{n,N} |D_{s_k} a_{ij}(x,s)\xi_i\xi_j| \leq 2e^{-4K}\psi'(|s|) \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j\,, \tag{1.12}$$

for almost every $x \in \Omega$ and for all $r, s \in \mathbb{R}^N$ and $\xi \in \mathbb{R}^n$.

The proof itself of [2, Lemma 6.1] shows that condition (1.12) implies our assumption (b). On the other hand, if $N > 1$, the two conditions look quite similar. However, (b) seems to be preferable, because when $N = 1$ it reduces to the inequality

$$\Big|\sum_{i,j=1}^{n} D_s a_{ij}(x,s)\xi_i\xi_j\Big| \leq 2\psi'(s) \sum_{i,j=1}^{n} a_{ij}(x,s)\xi_i\xi_j\,,$$

which is not so restrictive in view of (1.6), while (1.12) is in this case much stronger. We refer the reader to [2, examples 9.1–9.3] for some classes of coefficients fulfilling (1.12) and thus (b).

2. Symmetry-perturbed functionals

Given $\varphi \in L^2(\Omega, \mathbb{R}^N)$, we shall now consider the functional $f : H_0^1(\Omega, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$f(u) = \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx - \int_\Omega \varphi \cdot u \, dx.$$

If $\varphi \not\equiv 0$, clearly f is not even. Note that by (1.9) we find $c_1, c_2, c_3 > 0$ such that

$$\frac{1}{q}(s \cdot g(x,s) + c_1) \geq G(x,s) + c_2 \geq c_3 |s|^q. \tag{2.1}$$

Lemma 2.1. *Assume that $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to* (1.4). *Then there exists $\sigma > 0$ such that*

$$\int_\Omega (G(x,u) + c_2)\, dx \leq \sigma \left(f(u)^2 + 1\right)^{\frac{1}{2}}.$$

Proof. If $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to (1.4), taking into account (1.10), we deduce that

$$\begin{aligned} f(u) =& f(u) - \frac{1}{2} f'(u)(u) = \int_\Omega \left[\frac{1}{2} g(x,u) \cdot u - G(x,u) - \frac{1}{2}\varphi \cdot u\right] dx \\ & -\frac{1}{4}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u) \cdot u D_i u_h D_j u_h \, dx \\ \geq& \left(\frac{1}{2} - \frac{1}{q}\right) \int_\Omega (g(x,u)\cdot u + c_1)\, dx - \frac{1}{2}\|\varphi\|_2 \|u\|_2 \\ & -\frac{\gamma}{4}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - c_4 \\ \geq& \left(\frac{q}{2} - 1 - \frac{\gamma}{2}\right) \int_\Omega (G(x,u) + c_2)\, dx - \frac{\gamma}{2} f(u) - \varepsilon \|u\|_q^q - \beta(\varepsilon)\|\varphi\|_2^{q'} - c_5 \end{aligned}$$

with $\varepsilon \to 0$ and $\beta(\varepsilon) \to +\infty$. Choosing $\varepsilon > 0$ small enough, by (2.1) we have

$$\sigma f(u) \geq \int_\Omega (G(x,u) + c_2)\, dx - c_6,$$

where $\sigma = \frac{2+\gamma}{q-2-\gamma}$, and the assertion follows as in [19, Lemma 1.8]. □

We now want to introduce the modified functional, which is the main tool used to obtain our result. Let us define $\chi \in C^\infty(\mathbb{R})$ by setting $\chi = 1$ for

$s \leq 1$, $\chi = 0$ for $s \geq 2$ and $-2 < \chi' < 0$ when $1 < s < 2$, and let for each $u \in H_0^1(\Omega, \mathbb{R}^N)$

$$\phi(u) = 2\sigma \left(f(u)^2 + 1\right)^{\frac{1}{2}}, \qquad \psi(u) = \chi\Big(\phi(u)^{-1} \int_\Omega (G(x,u) + c_2)\; dx\Big).$$

Finally, we define the modified functional by

$$\widetilde{f}(u) = \tfrac{1}{2}\int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x,u) D_i u_h D_j u_h \, dx - \int_\Omega G(x,u)\, dx - \psi(u) \int_\Omega \varphi \cdot u \, dx. \tag{2.2}$$

The Euler's equation associated with the previous functional is given by

$$-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u) D_i u_k) + \tfrac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u) D_i u_h D_j u_h = \widetilde{g}(x,u) \tag{2.3}$$

in Ω, where we set $\widetilde{g}(x,u) = g(x,u) + \psi(u)\varphi + \psi'(u) \int_\Omega \varphi \cdot u \, dx$. Note that taking into account the previous lemma, if $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to (1.4), we have that $\psi(u) = 1$, and therefore $\widetilde{f}(u) = f(u)$. In the next result, we measure the defect of symmetry of $\widetilde{f}$, which turns out to be crucial in the final comparison argument.

Lemma 2.2. *There exists $\beta > 0$ such that for all $u \in H_0^1(\Omega, \mathbb{R}^N)$*

$$|\widetilde{f}(u) - \widetilde{f}(-u)| \leq \beta\big(|\widetilde{f}(u)|^{\frac{1}{q}} + 1\big).$$

Proof. Note first that if $u \in \operatorname{supt}(\psi)$, then

$$\Big| \int_\Omega \varphi \cdot u \, dx \Big| \leq \alpha(|f(u)|^{\frac{1}{q}} + 1), \tag{2.4}$$

where $\alpha > 0$ depends on $\|\varphi\|_2$. Indeed, by (2.1), we have

$$\Big| \int_\Omega \varphi \cdot u \, dx \Big| \leq \|u\|_2 \|\varphi\|_2 \leq c\|u\|_q \leq \hat{c}\Big(\int_\Omega (G(x,u) + c_2)\, dx\Big)^{\frac{1}{q}},$$

and since $u \in \operatorname{supt}(\psi)$,

$$\int_\Omega (G(x,u) + c_2)\; dx \leq 4\sigma \left(f(u)^2 + 1\right)^{\frac{1}{2}} \leq \tilde{c}(|f(u)| + 1),$$

inequality (2.4) easily follows. Now, since of course

$$|f(u)| \leq |\widetilde{f}(u)| + 2\Big| \int_\Omega \varphi \cdot u \, dx \Big|,$$

by (2.4) we immediately get for some $b > 0$

$$\psi(u)\Big|\int_\Omega \varphi\cdot u\,dx\Big| \le b\psi(u)\Big(|\widetilde{f}(u)|^{\frac{1}{q}} + \Big|\int_\Omega \varphi\cdot u\,dx\Big|^{\frac{1}{q}} + 1\Big).$$

Using Young's inequality, for some $b_1, b_2 > 0$, we have that

$$\psi(u)\Big|\int_\Omega \varphi\cdot u\,dx\Big| \le b_1(|\widetilde{f}(u)|^{\frac{1}{q}} + 1),$$

and

$$\psi(-u)\Big|\int_\Omega \varphi\cdot u\,dx\Big| \le b_2(|\widetilde{f}(u)|^{\frac{1}{q}} + 1),$$

and since

$$|\widetilde{f}(u) - \widetilde{f}(-u)| = (\psi(u) + \psi(-u))\Big|\int_\Omega \varphi\cdot u\,dx\Big|,$$

the assertion follows. $\square$

Theorem 2.3. *There exists $M > 0$ such that if $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to* (2.3) *with $\widetilde{f}(u) \ge M$ then u is a weak solution to* (1.4) *and $\widetilde{f}(u) = f(u)$.*

Proof. Let us first prove that there exist $\tilde{M} > 0$ and $\tilde{\alpha} > 0$ such that

$$\forall M \in [\tilde{M}, +\infty): \ \ \widetilde{f}(u) \ge M, \ \ u \in \operatorname{supt}(\psi) \Longrightarrow f(u) \ge \tilde{\alpha} M. \tag{2.5}$$

Since we have $f(u) \ge \widetilde{f}(u) - \big|\int_\Omega \varphi\cdot u\big|$, by (2.4) we deduce that

$$f(u) + \alpha\,|f(u)|^{\frac{1}{q}} \ge \widetilde{f}(u) - \alpha \ge \frac{M}{2} \ \text{ for } M \ge \tilde{M},$$

with $\tilde{M}$ large enough. Now, if it was $f(u) \le 0$, we would obtain

$$\frac{\alpha^{q'}}{q'} + \frac{1}{q}\,|f(u)| \ge \alpha\,|f(u)|^{\frac{1}{q}} \ge \frac{M}{2} + |f(u)|,$$

which is not possible if we take $\tilde{M} > 2\alpha^{q'}(q')^{-1}$. Therefore it is $f(u) > 0$ and $f(u) > \frac{M}{4}$ or $f(u) \ge \big(\frac{M}{4\alpha}\big)^q$, and (2.5) is proven. Of course, taking into account the definition of ψ, to prove the theorem it suffices to show that if $M > 0$ is sufficiently large and $u \in H_0^1(\Omega, \mathbb{R}^N)$ is a weak solution to (2.3) with $\widetilde{f}(u) \ge M$, then

$$\phi(u)^{-1}\int_\Omega (G(x,u) + c_2)\,dx \le 1.$$

If we set $\vartheta(u) = \phi(u)^{-1} \int_\Omega (G(x,u) + c_2)\, dx$, it follows that

$$\psi'(u)(u) = \chi'(\vartheta(u))\phi(u)^{-2}\Big[\phi(u)\int_\Omega g(x,u)\cdot u\,dx - (2\sigma)^2\vartheta(u)f(u)f'(u)(u)\Big].$$

Define now $T_1, T_2 : H_0^1(\Omega,\mathbb{R}^N) \to \mathbb{R}$ by setting

$$T_1(u) = \chi'(\vartheta(u))(2\sigma)^2\vartheta(u)\phi(u)^{-2}f(u)\int_\Omega \varphi\cdot u\,dx,$$

and

$$T_2(u) = \chi'(\vartheta(u))\phi(u)^{-1}\int_\Omega \varphi\cdot u\,dx + T_1(u).$$

Then we obtain

$$\begin{aligned}\widetilde{f}'(u)(u) &= (1+T_1(u))\int_\Omega \sum_{i,j=1}^n\sum_{h=1}^N a_{ij}^h(x,u)D_iu_hD_ju_h\,dx\\ &+ \frac{1}{2}(1+T_1(u))\int_\Omega \sum_{i,j=1}^n\sum_{h=1}^N D_sa_{ij}^h(x,u)\cdot uD_iu_hD_ju_h\,dx\\ &- (1+T_2(u))\int_\Omega g(x,u)\cdot u\,dx - (\psi(u)+T_1(u))\int_\Omega \varphi\cdot u\,dx.\end{aligned}$$

Consider now the term $\widetilde{f}(u) - \frac{1}{2(1+T_1(u))}\widetilde{f}'(u)(u)$. If $\psi(u) = 1$ and $T_1(u) = 0 = T_2(u)$, the assertion follows from Lemma 2.1. Otherwise, since $0 \le \psi(u) \le 1$, if $T_1(u)$ and $T_2(u)$ are both small enough the computations we have made in Lemma 2.1 still hold true with σ replaced by $(2-\varepsilon)\sigma$, for a small $\varepsilon > 0$, and again the assertion follows as in Lemma 2.1.

It then remains to show that if $M \to \infty$, then $T_1(u), T_2(u) \to 0$. We may assume that $u \in \mathrm{supt}(\psi)$; otherwise $T_i(u) = 0$, for $i = 1,2$. Therefore, taking into account (2.4), there exists $c > 0$ with

$$|T_1(u)| \le c\frac{|f(u)|^{\frac{1}{q}} + 1}{|f(u)|}.$$

Finally, by (2.5) we deduce $|T_1(u)| \to 0$ as $M \to \infty$. Similarly, $|T_2(u)| \to 0$.

3. Boundedness of concrete Palais–Smale sequences

Definition 3.1. Let $c \in \mathbb{R}$. A sequence $(u^m) \subseteq H_0^1(\Omega,\mathbb{R}^N)$ is said to be *a concrete Palais–Smale sequence at level c* (*$(CPS)_c$–sequence*, in short) for

$\widetilde{f}$, if $\widetilde{f}(u^m) \to c$,

$$\sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{s_k} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \in H^{-1}(\Omega, \mathbb{R}^N)$$

eventually as $m \to \infty$ and

$$-\sum_{i,j=1}^{n} D_j(a_{ij}^k(x,u^m) D_i u_k^m) + \tfrac{1}{2} \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_{s_k} a_{ij}^h(x,u^m) D_i u_h^m D_j u_h^m - \widetilde{g}_k(x, u^m) \to 0,$$

strongly in $H^{-1}(\Omega, \mathbb{R}^N)$, where $\widetilde{g}(x,u) = g(x,u) + \psi(u)\varphi + \psi'(u) \int_\Omega \varphi \cdot u \, dx$. We say that $\widetilde{f}$ satisfies *the concrete Palais–Smale condition at level* c, if every $(CPS)_c$ sequence for $\widetilde{f}$ admits a strongly convergent subsequence in $H_0^1(\Omega, \mathbb{R}^N)$.

Lemma 3.2. *There exists $M > 0$ such that each $(CPS)_c-$sequence (u^m) for $\widetilde{f}$ with $c \geq M$ is bounded in $H_0^1(\Omega, \mathbb{R}^N)$.*

Proof. Let $M > 0$ and (u^m) be a $(CPS)_c$-sequence for $\widetilde{f}$ with $c \geq M$ in $H_0^1(\Omega, \mathbb{R}^N)$ such that, eventually as $m \to +\infty$, $M \leq \widetilde{f}(u^m) \leq K$ for some $K > 0$. Taking into account [21, Lemma 3], we have $\lim_m \widetilde{f}'(u^m)(u^m) = 0$. Therefore, for large $m \in \mathbb{N}$ and any $\varrho > 0$, it follows that

$$\begin{aligned}
&\varrho \|u^m\|_{1,2} + K \geq \widetilde{f}(u^m) - \varrho \widetilde{f}'(u^m)(u^m) \\
&= \left(\tfrac{1}{2} - \varrho(1 + T_1(u^m))\right) \int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\
&- \frac{\varrho}{2}(1 + T_1(u^m)) \int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} D_s a_{ij}^h(x, u^m) \cdot u^m D_i u_h^m D_j u_h^m \, dx \\
&+ \varrho(1 + T_2(u^m)) \int_\Omega g(x, u^m) \cdot u^m \, dx \\
&- \int_\Omega G(x, u^m) \, dx + [\varrho(\psi(u^m) + T_1(u^m)) - \psi(u^m)] \int_\Omega \varphi \cdot u^m \, dx \\
&\geq \left(\tfrac{1}{2} - \varrho(1 + T_1(u^m)) - \frac{\varrho\gamma}{2}(1 + T_1(u^m))\right) \int_\Omega \sum_{i,j=1}^{n} \sum_{h=1}^{N} a_{ij}^h(x, u^m) D_i u_h^m D_j u_h^m \, dx \\
&+ \varrho(1 + T_2(u^m)) \int_\Omega g(x, u^m) \cdot u^m \, dx
\end{aligned}$$

$$
\begin{aligned}
&- \int_\Omega G(x,u^m)\,dx + [\varrho(\psi(u^m)+T_1(u^m)) - \psi(u^m)] \int_\Omega \varphi \cdot u^m\,dx \\
&\geq \frac{\nu}{2}\,(1-\varrho\,(2+\gamma)\,(1+T_1(u^m)))\,\|u^m\|_{1,2}^2 \\
&+ (q\varrho(1+T_2(u^m))-1)\int_\Omega G(x,u^m)\,dx - [\varrho(1+T_1(u^m))+1]\|\varphi\|_2\|u^m\|_2.
\end{aligned}
$$

If we choose M sufficiently large, we find $\varepsilon > 0$, $\eta > 0$ and $\varrho \in (\frac{1+\eta}{q}, \frac{1-\varepsilon}{\gamma+2})$ such that uniformly in $m \in \mathbb{N}$

$$
(1-\varrho\,(2+\gamma)\,(1+T_1(u^m))) > \varepsilon, \qquad (q\varrho(1+T_2(u^m))-1) > \eta.
$$

Hence we obtain

$$
\varrho\|u^m\|_{1,2} + K \geq \frac{\nu\varepsilon}{2}\|u^m\|_{1,2}^2 + b\eta\|u^m\|_q^q - c\|u^m\|_{1,2},
$$

which implies that the sequence (u^m) is bounded in $H_0^1(\Omega,\mathbb{R}^N)$. □

Lemma 3.3. *Let $c \in \mathbb{R}$. Then there exists $M > 0$ such that for each bounded $(CPS)_c$ sequence (u^m) for $\widetilde{f}$ with $c \geq M$, the sequence $(\widetilde{g}(x,u^m))$ admits a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$.*

Proof. Let (u^m) be a bounded $(CPS)_c$-sequence for $\widetilde{f}$ with $c \geq M$. We may assume that $(u^m) \subseteq \mathrm{supt}(\psi)$; otherwise $\psi(u^m) = 0$ and $\psi'(u^m) = 0$. Recall that

$$
\widetilde{g}(x,u^m) = g(x,u^m) + \psi(u^m)\varphi + \psi'(u^m)\int_\Omega \varphi \cdot u^m\,dx.
$$

Since by [9, Theorem 2.2.7] the maps from $H_0^1(\Omega,\mathbb{R}^N)$ to $H^{-1}(\Omega,\mathbb{R}^N)$, $u \longmapsto g(x,u)$ and $u \longmapsto \psi(u)\varphi$, are completely continuous, the sequences $(g(x,u^m))$ and $(\psi(u^m)\varphi)$ admit a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$. Now, we have

$$
\begin{aligned}
\psi'(u^m) = &\left[\chi'(\vartheta(u^m))\phi(u^m)^{-1}\right] g(x,u^m) \\
&- \left[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)\right] f'(u^m).
\end{aligned}
$$

On the other hand we have

$$
f'(u^m) = \widetilde{f}'(u^m) + \Big[\int_\Omega \varphi \cdot u^m\,dx\Big]\psi'(u^m) + [\psi(u^m)-1]\varphi.
$$

Therefore, we deduce that

$$
\begin{aligned}
&\Big[1 + \Big[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)\int_\Omega \varphi \cdot u^m\,dx\Big]\Big]\psi'(u^m) \\
&= \left[\chi'(\vartheta(u^m))\phi(u^m)^{-1}\right] g(x,u^m) \\
&- \left[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)\right] \widetilde{f}'(u^m)
\end{aligned} \tag{3.1}
$$

$$- \left[4\sigma^2\chi'(\vartheta(u^m))\phi(u^m)^{-2}\vartheta(u^m)f(u^m)(\psi(u^m)-1)\right]\varphi.$$

By assumption we have $\widetilde{f}'(u^m) \to 0$ in $H^{-1}(\Omega,\mathbb{R}^N)$. Taking into account the definition of χ, ϕ and ϑ, all of the square brackets in equation (3.1) are bounded in $\mathbb{R}$ for some $M>0$, and we conclude that also $(\psi'(u^m))$ admits a convergent subsequence in $H^{-1}(\Omega,\mathbb{R}^N)$. The assertion is now proven. $\square$

4. Compactness of concrete Palais–Smale sequences

The next result is the crucial property for the Palais–Smale condition to hold.

Lemma 4.1. *Let (u^m) be a bounded sequence in $H^1_0(\Omega,\mathbb{R}^N)$, and set*

$$\begin{aligned}\langle w^m, v\rangle &= \int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N a_{ij}^h(x,u^m) D_i u_h^m D_j v_h \, dx \\ &\quad + \frac{1}{2}\int_\Omega \sum_{i,j=1}^n \sum_{h=1}^N D_s a_{ij}^h(x,u^m)\cdot v D_i u_h^m D_j u_h^m \, dx\end{aligned}$$

for all $v \in C_c^\infty(\Omega,\mathbb{R}^N)$. Then, if (w^m) is strongly convergent to some w in $H^{-1}(\Omega,\mathbb{R}^N)$, (u^m) admits a strongly convergent subsequence in $H^1_0(\Omega,\mathbb{R}^N)$.

Proof. See, [21, Lemma 6]. $\square$

The next result is one of the main tools of this paper, the $(CPS)_c$ condition for $\widetilde{f}$.

Theorem 4.2. *There exists $M>0$ such that $\widetilde{f}$ satisfies the $(CPS)_c$-condition for $c \geq M$.*

Proof. Let (u^m) be a $(CPS)_c$-sequence for f with $c \geq M$, where $M>0$ is as in Lemma 3.2. Therefore, (u^m) is bounded in $H^1_0(\Omega,\mathbb{R}^N)$, and from Lemma 3.3 we deduce that, up to subsequences, $(\widetilde{g}(x,u^m))$ is strongly convergent in $H^{-1}(\Omega,\mathbb{R}^N)$. Therefore, the assertion follows from Lemma 4.1. $\square$

5. Existence of multiple solutions

Let (λ_h, u_h) be the sequence of eigenvalues and eigenvectors for the problem

$$\begin{cases}\Delta u = -\lambda u & \text{in } \Omega\\ u = 0 & \text{on } \partial\Omega,\end{cases}$$

and set $V_k = \overline{\mathrm{span}}\{u_1, \dots, u_k \in H_0^1(\Omega, \mathbb{R}^N)\}$. By (1.9) we deduce that for all $s \in \mathbb{R}^N$,

$$|s| \geq R \Longrightarrow G(x,s) \geq \frac{G(x, R\frac{s}{|s|})}{R^q}|s|^q \geq b_0(x)|s|^q,$$

where $b_0(x) = R^{-q} \inf\{G(x,s) : \ |s| = R\} > 0$. Then it follows that for each $k \in \mathbb{N}$ there exists $R_k > 0$ such that for all $u \in V_k$, $\|u\|_{1,2} \geq R_k$ implies $\widetilde{f}(u) \leq 0$. We can now give the following

Definition 5.1. For each $k \in \mathbb{N}$, set $D_k = V_k \cap B(0, R_k)$, $\Gamma_k = \{\gamma \in C(D_k, H_0^1) : \gamma \text{ odd and } \gamma_{|_{\partial B(0,R_k)}} = Id\}$, and $b_k = \inf_{\gamma \in \Gamma_k} \max_{u \in D_k} \widetilde{f}(\gamma(u))$.

Lemma 5.2. *For each $k \in \mathbb{N}$, $\varrho \in (0, R_k)$ and $\gamma \in \Gamma_k$, $\gamma(D_k) \cap \partial B(0, \varrho) \cap V_{k-1}^{\perp} \neq \emptyset$.*

Proof. See, [19, Lemma 1.44] . □

Lemma 5.3. *There exist $\beta > 0$ and $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, $b_k \geq \beta k^{\frac{(n+2)-(n-2)\sigma}{n(\sigma-1)}}$.*

Proof. Let $\gamma \in \Gamma_k$ and $\varrho \in (0, R_k)$. By the previous lemma there exists $w \in \gamma(D_k) \cap \partial B(0,\varrho) \cap V_{k-1}^{\perp}$, and therefore

$$\max_{u \in D_k} \widetilde{f}(\gamma(u)) \geq \widetilde{f}(w) \geq \inf_{u \in \partial B(0,\varrho) \cap V_{k-1}^{\perp}} \widetilde{f}(u). \tag{5.2}$$

Given $u \in \partial B(0,\varrho) \cap V_{k-1}^{\perp}$, by (1.11) we find $\alpha_1, \alpha_2, \alpha_3 > 0$ with

$$\widetilde{f}(u) \geq \frac{1}{2}\varrho^2 - \alpha_1 \|u\|_{\sigma+1}^{\sigma+1} - \alpha_2 \|\varphi\|_2 \|u\|_2 - \alpha_3.$$

Now, by the Gagliardo-Niremberg inequality, there is $\alpha_4 > 0$ such that

$$\|u\|_{\sigma+1} \leq \alpha_4 \|u\|_{1,2}^{\vartheta} \|u\|_2^{1-\vartheta},$$

where $\vartheta = \frac{n(\sigma-1)}{2(\sigma+1)}$. As is well known, $\|u\|_2 \leq \frac{1}{\sqrt{\lambda_{k-1}}}\|u\|_{1,2}$, so that we obtain

$$\widetilde{f}(u) \geq \frac{1}{2}\varrho^2 - \alpha_1 \lambda_k^{-\frac{(1-\vartheta)(\sigma+1)}{2}} \varrho^{\sigma+1} - \alpha_2 \|\varphi\|_2 \lambda_k^{-\frac{1}{2}} \varrho - \alpha_3.$$

Choosing now $\varrho = c\lambda_k^{-\frac{(1-\vartheta)}{2}\frac{(\sigma+1)}{(\sigma-1)}}$ yields $\widetilde{f}(u) \geq \frac{1}{4}\varrho_k^2 - \alpha_2\|\varphi\|_2 \lambda_k^{-\frac{1}{2}}\varrho_k - \alpha_3$. Now, as is shown in [11], there exists $\alpha_5 > 0$ such that for large k, $\lambda_k \geq \alpha_5 k^{\frac{2}{n}}$. Therefore, we find $\beta > 0$ with $\widetilde{f}(u) \geq \beta k^{\frac{(n+2)-(n-2)\sigma}{n(\sigma-1)}}$, and by (5.2) the lemma is proved. □

Definition 5.4. For each $k \in \mathbb{N}$, set $U_k = \{\xi = tu_{k+1} + w : t \in [0, R_{k+1}], w \in B(0, R_{k+1}) \cap V_k, \|\xi\|_{1,2} \leq R_{k+1}\}$ and $\Lambda_k = \{\lambda \in C(U_k, H^1_0) : \lambda_{|_{D_k}} \in \Gamma_{k+1}$ and $\lambda|_{\partial B(0,R_{k+1}) \cup ((B(0,R_{k+1}) \setminus B(0,R_k)) \cap V_k)} = Id\}$ and $c_k = \inf_{\lambda \in \Lambda_k} \max_{u \in U_k} \widetilde{f}(\lambda(u))$.

We now come to our main existence tool. Of course, differently from the proof of [19, Lemma 1.57], in this nonsmooth framework, we shall apply [9, Theorem 1.1.13] instead of the classical deformation lemma [19, Lemma 1.60].

Lemma 5.5. *Assume that $c_k > b_k \geq M$, where M is as in Theorem 4.2. If $\delta \in (0, c_k - b_k)$ and $\Lambda_k(\delta) = \{\lambda \in \Lambda_k : \widetilde{f}(\lambda(u)) \leq b_k + \delta$ for $u \in D_k\}$, set*

$$c_k(\delta) = \inf_{\lambda \in \Lambda_k(\delta)} \max_{u \in U_k} \widetilde{f}(\lambda(u)) \,.$$

Then $c_k(\delta)$ is a critical value for $\widetilde{f}$.

Proof. Let $\overline{\varepsilon} = \frac{1}{2}(c_k - b_k - \delta) > 0$, and assume for the sake of contradiction that $c_k(\delta)$ is not a critical value for $\widetilde{f}$. Therefore, taking into account Lemma 4.2, by [9, Theorem 1.1.13], there exists $\varepsilon > 0$ and a continuous map $\eta : H^1_0(\Omega, \mathbb{R}^N) \times [0,1] \to H^1_0(\Omega, \mathbb{R}^N)$ such that for each $u \in H^1_0(\Omega, \mathbb{R}^N)$ and $t \in [0,1]$

$$\widetilde{f}(u) \notin (c_k(\delta) - \overline{\varepsilon}, c_k(\delta) + \overline{\varepsilon}) \implies \eta(u,t) = u, \tag{5.3}$$

and

$$\eta(\widetilde{f}^{c_k(\delta)+\varepsilon}, 1) \subseteq \widetilde{f}^{c_k(\delta)-\varepsilon}. \tag{5.4}$$

Choose $\lambda \in \Lambda_k(\delta)$ so that

$$\max_{u \in U_k} \widetilde{f}(\lambda(u)) \leq c_k(\delta) + \varepsilon, \tag{5.5}$$

and consider $\eta(\lambda(\cdot), 1) : U_k \to H^1_0(\Omega, \mathbb{R}^N)$. Observe that if $u \in \partial B(0, R_{k+1})$ or $u \in (B(0, R_{k+1}) \setminus B(0, R_k)) \cap V_k$, by definition $\widetilde{f}(\lambda(u)) = \widetilde{f}(u)$. Hence, by (5.3), we have $\eta(\lambda(u), 1) = u$. We conclude that $\eta(\lambda(\cdot), 1) \in \Lambda_k$. Moreover, by our choice of $\overline{\varepsilon} > 0$ and $\delta > 0$ we obtain $\forall u \in D_k$, $\widetilde{f}(\lambda(u)) \leq b_k + \delta \leq c_k - \overline{\varepsilon} \leq c_k(\delta) - \overline{\varepsilon}$. Therefore, (5.3) implies that $\eta(\lambda(\cdot), 1) \in \Lambda_k(\delta)$. On the other hand, again by (5.4) and (5.5)

$$\max_{u \in U_k} \widetilde{f}(\eta(\lambda(u), 1)) \leq c_k(\delta) - \varepsilon, \tag{5.6}$$

which is not possible, by definition of $c_k(\delta)$. □

It only remains to prove that we cannot have $c_k = b_k$ for k sufficiently large.

Lemma 5.6. *Assume that $c_k = b_k$ for all $k \geq k_1$. Then, there exist $\gamma > 0$ and $\tilde{k} \geq k_1$ with $b_{\tilde{k}} \leq \gamma \tilde{k}^{\frac{q}{q-1}}$.*

Proof. Choose $k \geq k_1$, $\varepsilon > 0$ and a $\lambda \in \Lambda_k$ such that $\max_{u \in U_k} \widetilde{f}(\lambda(u)) \leq b_k + \varepsilon$. Define now $\widetilde{\lambda} : D_{k+1} \to H_0^1$ such that

$$\widetilde{\lambda}(u) = \begin{cases} \lambda(u) & \text{if } u \in \ \ U_k \\ -\lambda(-u) & \text{if } u \in -U_k. \end{cases}$$

Since $\widetilde{\lambda}_{|_{B(0,R_{k+1}) \cap V_k}}$ is continuous and odd, it follows that $\widetilde{\lambda} \in \Gamma_{k+1}$. Then $b_{k+1} \leq \max_{u \in D_{k+1}} \widetilde{f}(\widetilde{\lambda}(u))$. By Lemma 2.2, we have $\max_{u \in -U_k} \widetilde{f}(\widetilde{\lambda}(u)) \leq b_k + \varepsilon + \beta(|b_k + \varepsilon|^{\frac{1}{q}} + 1)$, and since $D_{k+1} = U_k \cup (-U_k)$, we get $\forall \varepsilon > 0$,

$$b_{k+1} \leq b_k + \varepsilon + \beta(|b_k + \varepsilon|^{\frac{1}{q}} + 1),$$

which yields $\forall k \geq k_1$, $b_{k+1} \leq b_k + \beta(|b_k|^{\frac{1}{q}} + 1)$. The assertion now follows recursively as in [20, Proposition 10.46]. □

We finally come to the proof of the main result, which extends the theorems of [3, 14, 19, 23] to the quasilinear case, both scalar and vectorial.

Proof of Theorem 1.1. Observe that the inequality

$$1 < \sigma < \frac{qn + (q-1)(n+2)}{qn + (q-1)(n-2)} \quad \text{implies} \quad \frac{q}{q-1} < \frac{(n+2) - \sigma(n-2)}{n(\sigma - 1)}.$$

Therefore, combining Lemma 5.3 and Lemma 5.6 we deduce $c_k > b_k$ so that we may apply Lemma 5.5 and obtain that $(c_k(\delta))$ is a sequence of critical values for $\widetilde{f}$. By Theorem 2.3 we finally conclude that f has a diverging sequence of critical values. □

Remark 5.7. In 1988 and 1992, A. Bahri and P. L. Lions showed via a perturbation technique based on Morse theory that, at least in some particular cases, the growth restriction on σ is not essential. More precisely, they proved that the problem

$$-\Delta u = |u|^{\sigma - 1} u - \varphi \quad \text{in } \Omega$$

has a sequence (u_h) of solutions in $H_0^1(\Omega)$ for each $\sigma \in (1, \frac{n+2}{n-2})$ (see [4, 5]).

One knows from Pohozaev's identity that even when $\varphi \equiv 0$ this result is false in general if $\sigma > \frac{n+2}{n-2}$ so that this theorem seems to be optimal. The problem of whether or not this existence result holds also in the quasilinear case is open.

Remark 5.8. In this paper we treat only existence of weak solutions. For regularity results in the scalar case, we refer the reader to [9]. For regularity results in the vectorial quasilinear case, we refer to [16, 21] and the references therein.

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