

Test of Dynamical Systems in Finance - Solution - January 14, 2022

1. By imposing $D_{h+1} = S_{h+1}$ one has

$$p_{h+1} = -2p_h + 1$$

and the equilibrium position is given by

$$-2\bar{p} + 1 = \bar{p} \Rightarrow \bar{p} = \frac{1}{3}.$$

Since the derivative of the r.h.s. is -2, which is larger than 1 in absolute value, the position is unstable. The explicit solution is given by

$$p_h = (p_0 - \bar{p})(-2)^h + \bar{p} = \frac{2}{3}(-2)^h + \frac{1}{3},$$

hence $p_4 = \frac{2}{3}(-2)^4 + \frac{1}{3} = \frac{32}{3} + \frac{1}{3} = 11$.

2. Solving the system of the equilibria we get

$$\begin{cases} P_1(1 - P_2) = 0 \\ P_2(P_1 + k) = 0 \end{cases} \Rightarrow \begin{cases} P_1 = 0 \\ P_2 = 0 \end{cases} \text{ or } \begin{cases} P_1 = -k \\ P_2 = 1. \end{cases}$$

The matrix of the derivatives is

$$A = \begin{bmatrix} 1 - P_2 & -P_1 \\ P_2 & P_1 + k \end{bmatrix} \Rightarrow A(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix} \text{ and } A(-k, 1) = \begin{bmatrix} 0 & k \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix $A(0, 0)$ are strictly positive, hence the position is an unstable node. Since $k > 0$, the eigenvalues of $A(-k, 1)$ are given by

$$\lambda^2 - k = 0 \Rightarrow \lambda_{1,2} = \pm\sqrt{k}$$

hence also the second position is unstable (saddle).

3. Since $q \neq \frac{1}{2}$, the formula for the probability of the gambler's ruin is given by

$$p_h = \frac{x^h - x^N}{1 - x^N}, \quad x = \frac{1 - q}{q}$$

where q is the probability of winning a single play and N is the target. Hence in this case $x = \frac{1}{2}$ and

$$p_2 = \frac{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^5}{1 - \left(\frac{1}{2}\right)^5} = \frac{\frac{1}{4} - \frac{1}{32}}{1 - \frac{1}{32}} = \frac{7}{31} \simeq 0.226.$$

4. By setting $Z = \dot{G}$ and deriving the last equation we have

$$\begin{cases} \dot{Y} = -\frac{2}{3}Y + G - 2 \\ \dot{G} = Z \\ \dot{Z} = -Z - Y \end{cases}$$

which has the equilibrium position $Y = 0, G = 2, Z = 0$. For studying the stability we write the matrix of the linear system:

$$\begin{bmatrix} -2/3 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}$$

which has characteristic polynomial

$$\lambda^3 + \frac{5}{3}\lambda^2 + \frac{2}{3}\lambda + 1,$$

where all the coefficients are positive. The Hurwitz matrix writes

$$\begin{bmatrix} 5/3 & 1 & 0 \\ 1 & 2/3 & 0 \\ 0 & 5/3 & 1 \end{bmatrix}$$

and by the Liénard-Chipart criterion it is sufficient to compute $\Delta_2 = \frac{10}{9} - 1 > 0$, hence the equilibrium is asymptotically stable.

5. The equation for the equilibrium is

$$x - \frac{8}{x} = x \quad \Rightarrow \quad \frac{8}{x} = 0$$

which has no solution. In order to look for 2-cycles, we have to solve

$$f(f(x)) = x \text{ where } f(x) = x - \frac{8}{x},$$

hence

$$x - \frac{8}{x} - \frac{8}{x - \frac{8}{x}} = x \quad \Rightarrow \quad x - \frac{8}{x} = -x \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$$

hence there is one and only one 2-cycle: $\{-2, 2\}$.

In the general case $x_{h+1} = x_h - \frac{d}{x_h}$, the computations are similar: there are no equilibria and the 2-cycle is given by

$$x^2 = \frac{d}{2} \quad \Rightarrow \quad x = \pm \sqrt{\frac{d}{2}}$$

which is well defined for $d > 0$.