

UNIVERSITÀ CATTOLICA DEL SACRO CUORE

Dynamical Systems in Finance **December 11th, 2023**

1. The dynamics of a price in a Goodwin model is described by the equation

$$3p_{h+2} + (\alpha + 1)p_{h+1} - \alpha p_h = 4, \quad \alpha > 0.$$

Find the equilibrium position and discuss its stability.

In the case $\alpha = 1$ discuss the stability and write the explicit solutions.

2. A gambler starts from 1€ and has a target of winning 5€. If the probability of winning is $2/3$, it is more likely that the gambler get ruined or that he/she reaches the target?

What happens if the probability of winning drops to $1/2$?

3. Discuss the stability of all the equilibria of the nonlinear system of ODEs

$$\begin{cases} \dot{x} = y^2 \\ \dot{y} = \alpha x + y - z^2 \\ \dot{z} = x - z^2 \end{cases}$$

for $\alpha \in \mathbb{R}$.

4. A Phillips' stabilization model with integral policy is of the form

$$\begin{cases} \dot{Y} = -\frac{2}{5}Y + 2G - 2 \\ \dot{G} = -2G - k \int_0^t Y(\tau) d\tau \end{cases}$$

for $k > 0$. Write the equilibrium position and discuss its stability.

5. Discuss the stability of the equilibria of the nonlinear discrete equation

$$x_{h+2} - \frac{1}{2}x_{h+1} - x_{h+1}x_h - \frac{1}{4}x_h = 0.$$

$$1) \quad 3\bar{p} + (\alpha+1)\bar{p} - \alpha\bar{p} = 4 \quad 4\bar{p} = 4 \quad \bar{p} = 1$$

$$3\lambda^2 + (\alpha+1)\lambda - \alpha = 0$$

$$\lambda^2 + \frac{\alpha+1}{3}\lambda - \frac{\alpha}{3} = 0$$

$$\frac{\alpha+1}{3} < 1 - \frac{\alpha}{3} < 2$$

$$\begin{cases} \frac{\alpha+1}{3} < 1 - \frac{\alpha}{3} \Rightarrow \alpha+1 < 3-\alpha \Rightarrow 2\alpha < 2 \Rightarrow \boxed{\alpha < 1} & \text{G.A.S.} \\ 1 - \frac{\alpha}{3} < 2 \Rightarrow 3-\alpha < 6 \Rightarrow \alpha > -3 \checkmark \end{cases}$$

$$\boxed{\alpha < 1}$$

$\alpha > 1$ UNST.

$$\alpha = 1$$

$$3\lambda^2 + 2\lambda - 1 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1+3}}{3} = \frac{-1 \pm 2}{3} = \begin{cases} -1 \\ \frac{1}{3} \end{cases}$$

$$p_h = c_1 (-1)^h + c_2 \left(\frac{1}{3}\right)^h \quad \text{SIMPLY STABLE.}$$

$$2) \quad x = \frac{1 - \frac{2}{3}}{\frac{2}{3}} = \frac{1}{3} \cdot \frac{3}{2} = \frac{1}{2}$$

$$p_1 = \frac{x^4 - x^5}{1 - x^5} = \frac{\frac{1}{2} - \frac{1}{32}}{1 - \frac{1}{32}} = \frac{15}{31} < \frac{1}{2}$$

WIN > RUIN

$$\phi = \frac{1}{2} \Rightarrow p_1 = 1 - \frac{1}{5} = \frac{4}{5} \quad \text{RUIN > WIN}$$

$$3) \begin{cases} y^2 = 0 \Rightarrow y = 0 \\ \lambda x + y - z^2 = 0 \\ x - z^2 = 0 \Rightarrow x = z^2 \end{cases} \quad \begin{matrix} \lambda z^2 - z^2 = 0 \\ \lambda = 1 \quad \forall z \\ \lambda \neq 1 \quad z = 0 \end{matrix}$$

$$\lambda \neq 1 \quad (0, 0, 0)$$

$$\lambda = 1 \quad (\bar{z}^2, 0, \bar{z})$$

$$\begin{bmatrix} 0 & 0 & 2y & 0 \\ 2 & 1 & -2z \\ 1 & 0 & -2\bar{z} \end{bmatrix}$$

$$\lambda \neq 1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \lambda & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\lambda = 1 \quad \text{UNST.}$$

$$\lambda = 1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & -2\bar{z} \\ 1 & 0 & -2\bar{z} \end{bmatrix}$$

$$\lambda = 0$$

$$\lambda = 1$$

$$\lambda = -2\bar{z} \quad \text{UNST.}$$

$$4) \begin{cases} \dot{y} = -\frac{2}{5}y + 2G - 2 \\ \dot{G} = z \\ \dot{z} = -ky - 2z \end{cases}$$

$$G = 1$$

$$z = 0$$

$$y = 0$$

$$\det \begin{bmatrix} -\frac{2}{5} - \lambda & 2 & 0 \\ 0 & 0 - \lambda & 1 \\ -k & 0 & -2 - \lambda \end{bmatrix} = \lambda \left(\lambda + \frac{2}{5} \right) (-\lambda - 2) - 2k = \left(\lambda^2 + \frac{2}{5}\lambda \right) (-\lambda - 2) - 2k$$

$$= -\lambda^3 - 2\lambda^2 - \frac{2}{5}\lambda^2 - \frac{4}{5}\lambda - 2k$$

$$= \lambda^3 + \frac{12}{5}\lambda^2 + \frac{4}{5}\lambda + 2k$$

$$\begin{bmatrix} -\frac{12}{5} & 2k & 0 \\ 1 & \frac{4}{5} & 0 \\ 0 & \frac{12}{5} & 2k \end{bmatrix}$$

$$\frac{48}{25} - 2k > 0$$

$$k < \frac{24}{25} \quad \text{G.A.S.}$$

5)

$$\bar{x} - \frac{1}{2}\bar{x} - \bar{x}^2 - \frac{1}{4}\bar{x} = 0$$

$$\bar{x} = 0 \quad , \quad 1 - \frac{1}{2} - \frac{1}{4} = \bar{x}^0$$

$$\bar{x} = \frac{1}{4}$$

$$\begin{cases} x_{h+1} = y_h \\ y_{h+1} = \frac{1}{4}x_h + x_h y_h + \frac{1}{2}y_h \end{cases}$$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{4} + y & x + \frac{1}{2} \end{bmatrix}$$

(0,0)

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

$$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{4}$$

$$\left| \frac{1}{2} \right| < 1 - \frac{1}{4} < 2$$

$$\frac{1}{2} < \frac{3}{4} \quad \checkmark \quad \text{A.S.}$$

 $(\frac{1}{4}, \frac{1}{4})$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

$$\lambda^2 - \frac{3}{4}\lambda - \frac{1}{2}$$

$$\left| \frac{3}{4} \right| < 1 - \frac{1}{2} < 2$$

$$\frac{3}{4} < \frac{1}{2} \quad \text{UNST.}$$