ISTITUZIONI DI GEOMETRIA SUPERIORE

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① In $\mathbb{R}^2$, dati $X = y\frac{\partial}{\partial y}$, $T = \frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y} \otimes dy$

si calcoli $\mathcal{L}_X T$

② In $(0,2\pi) \times (0,\pi)$, dati $g = \sin^2\theta \, d\varphi^2 + d\theta^2$

e $X = f(\theta)\frac{\partial}{\partial \theta}$ ($f = f(\theta)$ liscia), determinare

f in modo che X sia di Killing.

Il risultato ottenuto era da attendersi?

③ Dati $(\mathbb{R}^2, \omega = dq \wedge dp)$ e $H = q^8 + p^8$,

struttura simplettica

determinare X_H (gradiente simplettico ≡ campo vett. hamiltoniano associato ad H)

Trovare le curve integrali di X_H. Si dimostri altresì che, rispetto alla metrica standard su $\mathbb{R}^2$, ∇H (gradiente riemanniano di H) è perpendicolare ad X_H pto per pto.

Tempo a disposizione: 1h

Le risposte vanno adeguatamente giustificate

Ishigroup
19/6/19

① in $\mathbb{R}^2$: $X = y\frac{\partial}{\partial y}$

$$T = \frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y} \otimes dy$$

Calcolare $\mathcal{L}_X T$

$$\mathcal{L}_X T \underset{\text{(Leibniz)}}{=} \boxed{\mathcal{L}_X\left(\frac{\partial}{\partial x}\right)} \otimes y\frac{\partial}{\partial y} \otimes dy + \frac{\partial}{\partial x} \otimes \boxed{\mathcal{L}_X\left(y\frac{\partial}{\partial y}\right)} \otimes dy + \frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y} \otimes \boxed{\mathcal{L}_X dy}$$

$$\mathcal{L}_{y\frac{\partial}{\partial y}}\left(\frac{\partial}{\partial x}\right) = \left[y\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right] = 0$$

$$\mathcal{L}_{y\frac{\partial}{\partial y}}\left(y\frac{\partial}{\partial y}\right) = \left[y\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}\right] = 0$$

$$\mathcal{L}_X dy = d\mathcal{L}_X y = d\big(X(y)\big) = d\left(y\frac{\partial}{\partial y}y\right) = $$
$$= dy$$

In definitiva $\mathcal{L}_X T = \frac{\partial}{\partial x} \otimes y\frac{\partial}{\partial y} \otimes dy = T$

Isti geo sup
19/6/19

② In $(0,2\pi)\times(0,\pi)$

data $g = \sin^2\theta \, d\varphi^2 + d\theta^2$

e dato $X = f(\theta)\frac{\partial}{\partial\theta}$ per quali f (lisce)

X è Killing?

$$\mathcal{L}_X g = \boxed{\mathcal{L}_X(\sin^2\theta)}\, d\varphi^2 + \sin^2\theta \, \mathcal{L}_X(d\varphi)^2 + \mathcal{L}_X(d\theta)^2$$

$$\mathcal{L}_X(\sin^2\theta) = f(\theta)\frac{\partial \sin^2\theta}{\partial\theta} = f(\theta)\, 2\sin\theta\cos\theta = f(\theta)\sin 2\theta$$

$$= \mathcal{L}_X d\varphi \, d\varphi + d\varphi \, \mathcal{L}_X d\varphi \; ; \quad \mathcal{L}_X d\varphi = d\mathcal{L}_X\varphi = d\, X(\varphi) = 0$$

$$= \mathcal{L}_X d\theta \, d\theta + d\theta \, \mathcal{L}_X d\theta \; ; \quad \mathcal{L}_X d\theta = d\mathcal{L}_X\theta = d\, f\cdot\frac{\partial\theta}{\partial\theta} = df = f'\, d\theta$$

$' = \frac{d}{d\theta}$

Dunque $\boxed{\mathcal{L}_X g = 0}$ (X Killing)

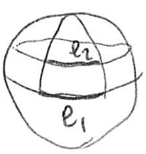
$\iff$

$$f(\theta)\sin 2\theta \, d\varphi^2 + 2f'\, d\theta^2 = 0$$

$$\Rightarrow \quad f\sin 2\theta = 0 \quad f' = 0 \quad \Rightarrow \quad f = 0 \quad \Rightarrow X = 0$$

Il risultato era da attendersi

$f\frac{\partial}{\partial\theta}$, ad esempio, $f\neq 0$ manda paralleli in paralleli, e dunque modifica le lunghezze.



$\ell_1 > \ell_2$ ..

(3) Dato $(\mathbb{R}^2, \omega = dq \wedge dp)$

e $H = q^8 + p^8$ trovare X_H (gradiente simplettico di H)

verificare direttamente che $\mathcal{L}_{X_H}\omega = 0$

Trovare le curve integrali di X_H e si provi che, rispetto alla metrica standard, ∇H (grad. riem) $\perp X_H$ punto per punto

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Omega^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\Omega$$

$$\Omega^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\Omega^{-T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \Omega$$

Sol curve integrali: $q^8 + p^8 = c$

$$i_{X_H}\omega = dH \qquad X_H = \Omega^{-T} \cdot \nabla H = \Omega \cdot \nabla H$$

$$\langle X_H, \nabla H \rangle = \langle \Omega \overset{X}{\nabla H}, \overset{Y}{\nabla H} \rangle = 0 \quad \text{poiché } \Omega^T = -\Omega:$$

$$\omega_{ij} y^i y^j = \omega_{ji} y^j y^i = \omega_{ji} y^i y^j = -\omega_{ij} y^i y^j$$

$$\Rightarrow \omega_{ij} y^i y^j = 0$$

$$\nabla H = 8q^7 \frac{\partial}{\partial q} + 8p^7 \frac{\partial}{\partial p}$$

$$X_H \Rightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 8q^7 \\ 8p^7 \end{pmatrix} = \begin{pmatrix} 8p^7 \\ -8q^7 \end{pmatrix}$$

$$X_H = 8p^7 \frac{\partial}{\partial q} - 8q^7 \frac{\partial}{\partial p} \qquad \nabla H = 8q^7 \frac{\partial}{\partial q} + 8p^7 \frac{\partial}{\partial p}$$

($X_H \perp \nabla H$, di nuovo)

Calcoliamo $\mathcal{L}_{X_H} \omega = 0$ direttamente

$$\mathcal{L}_{X_H}(dq \wedge dp) = \mathcal{L}_{X_H}(dq) \wedge dp + dq \wedge \mathcal{L}_{X_H} dp$$

$$= d\mathcal{L}_{X_H} q \wedge dp + dq \wedge d\mathcal{L}_{X_H} p$$

$$= d\, X_H(q) \wedge dp + dq \wedge d\, X_H(p)$$

$$= \underbrace{d\,(8p^7) \wedge dp}_{\shortparallel \atop 0} + \underbrace{dq \wedge d(-8q^7)}_{\shortparallel \atop 0}$$